The efficiency of resource allocation mechanisms for budget-constrained users

Ioannis Caragiannis **Alexandros A. Voudouris**

University of Patras

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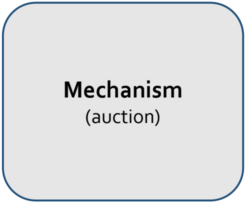

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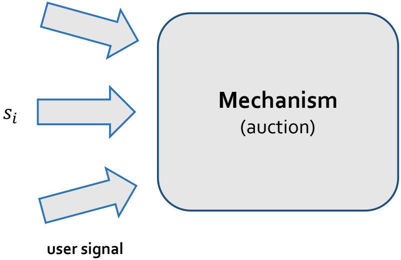

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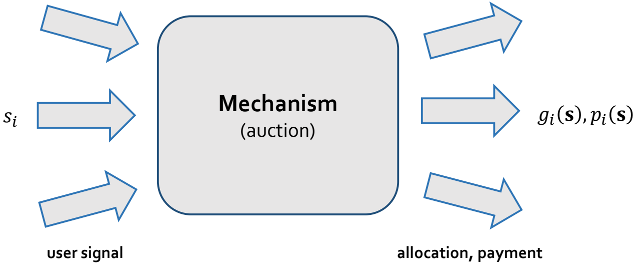

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- SH mechanism [Sanghavi and Hajek 2004]:

$$g_i(\mathbf{s}) = \frac{s_i}{\max_\ell s_\ell} \int_0^1 \prod_{j \neq i} \left(1 - \frac{s_j}{\max_\ell s_\ell} t\right) \, \mathrm{d}t$$

User characteristics and selfishness

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Each user i has

- a **valuation function** $v_i : [0,1] \to \mathbb{R}_{\geq 0}$ that represents her value for fractions of the resource (increasing, differentiable, concave)
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Strategic behavior

- Users are utility-maximizers
- Select s_i in order to maximize utility: $u_i(\mathbf{s}) = v_i(g_i(\mathbf{s})) - p_i(\mathbf{s})$
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Equilibrium

- A signal vector $\mathbf{s}$ for which all players simultaneously maximize their personal utilities

Efficiency without budgets

- The **social welfare** of an allocation $\mathbf{d}$ in game $\mathcal{G}$ is

$$\mathsf{SW}(\mathbf{d}, \mathcal{G}) = \sum_i v_i(d_i)$$

- Price of anarchy of mechanism M,

$$\mathsf{PoA}(M) = \sup_{\mathcal{G} \in M} \sup_{\mathbf{s} \in \mathsf{EQ}(\mathcal{G})} \frac{\max_{\mathbf{d}} \mathsf{SW}(\mathbf{d}, \mathcal{G})}{\mathsf{SW}(g(\mathbf{s}), \mathcal{G})}$$

Efficiency without budgets

- PoA$(\text{Kelly}) = 4/3$ [Johari and Tsitsiklis, 2004]
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- There exist mechanisms that achieve full efficiency: MB mechanisms [Maheswaran and Basar, 2006]

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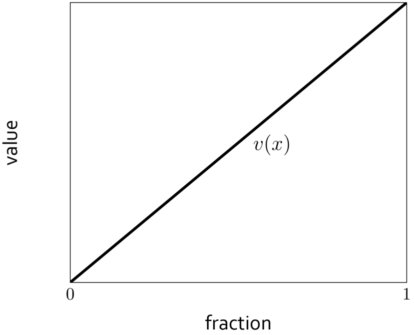

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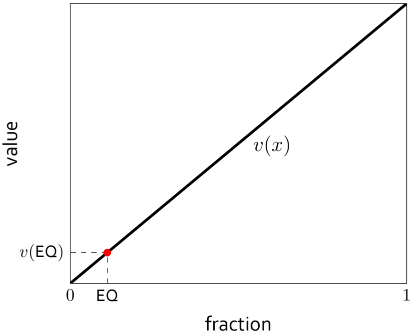

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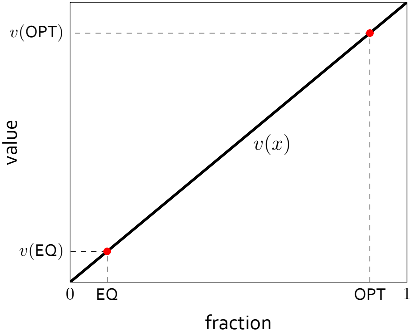

Efficiency with budgets

- Alternative: **liquid welfare** [Dobzinski and Paes Leme, 2014]
- The liquid welfare of an allocation $\mathbf{d}$ in game $\mathcal{G}$ is

$$\mathsf{LW}(\mathbf{d}, \mathcal{G}) = \sum_i \min\{v_i(d_i), c_i\}$$

- Liquid price of anarchy of mechanism M,

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- LW$(\mathbf{d}, \tilde{\mathcal{G}}) = d_1 + d_2 = 1$ vs. OPT $\geq 1 + d_2 \Rightarrow$ LPoA$(\tilde{\mathcal{G}}) \geq \frac{3}{2}$

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Two players

- LPoA$($E2-PYS$) = 1.792$ (best possible among all pay-your-signal mechanisms with concave allocation functions)
- LPoA$($E2-SR$) \leq 1.529$ (almost optimal over all two-player mechanisms)

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- Characterization:

$$\frac{p_2(\mathbf{s}) + \lambda_1(\mathbf{s})}{p_2(\mathbf{s}) + \lambda_1(\mathbf{s}) g_1(\mathbf{s})} = \frac{s_2 + \frac{s_2}{f'(z)}}{s_2 + \frac{s_2}{f'(z)} f(z)} = \frac{f'(z) + 1}{f'(z) + f(z)}$$

Two players: the design of E2-PYS

- Require:

$$\frac{f'(z)+1}{f'(z)+f(z)} = \beta \Leftrightarrow f'(z) + \frac{\beta}{\beta-1}f(z) = \frac{1}{\beta-1}$$

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- Definition of allocation function:

$$g_i(\mathbf{s}) = \begin{cases} \frac{1}{\beta} - \frac{1}{\beta}\exp\left(-\frac{\beta}{\beta-1}\frac{s_i}{s_{3-i}}\right) & s_i \leq s_{3-i} \\ \frac{\beta-1}{\beta} + \frac{1}{\beta}\exp\left(-\frac{\beta}{\beta-1}\frac{s_{3-i}}{s_i}\right) & s_i > s_{3-i} \end{cases}$$

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- By anonymity

$$f(1) = \frac{1}{2} \Leftrightarrow \frac{1}{\beta} - \frac{1}{\beta} \exp\left(-\frac{\beta}{\beta - 1}\right) = \frac{1}{2}$$

$$\Leftrightarrow \beta \approx 1.792$$

The design of E2-SR

- Change the payment function: $p_1(\mathbf{s}) = \frac{s_1}{s_2}$ and $p_2(\mathbf{s}) = \frac{s_2}{s_1}$
- Follow the same reasoning as with E2-PYS

Open problems

- Is there a simple mechanism that achieves the $2 - \frac{1}{n}$ bound?
- LPoA bounds over more general equilibrium concepts?
- Other resource allocation problems with budget constraints?
- Other efficiency benchmarks?

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Thank you!