The efficiency of resource allocation mechanisms for budget-constrained users

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Resource allocation

- One **divisible resource**
  - Bandwidth of a communication link
  - Processing time of a CPU
  - Storage space of a cloud

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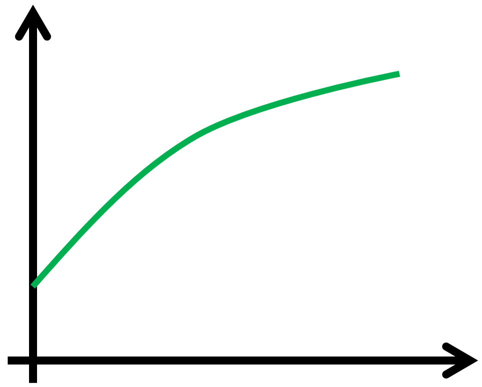
- n users such that user i has a **valuation function** $v_i: [0,1] \to \mathbb{R}_{\geq 0}$
  - $v_i(x)$ represents the value of user i for a fraction x of the resource
  - concave
  - non-decreasing
  - (semi-)differentiable



Resource allocation

Find an **allocation** $\boldsymbol{x} = (x_1, \ldots, x_n)$: $\sum_i x_i = 1$

to **maximize social welfare** $\mathrm{SW}(\boldsymbol{x}) = \sum_i v_i(x_i)$

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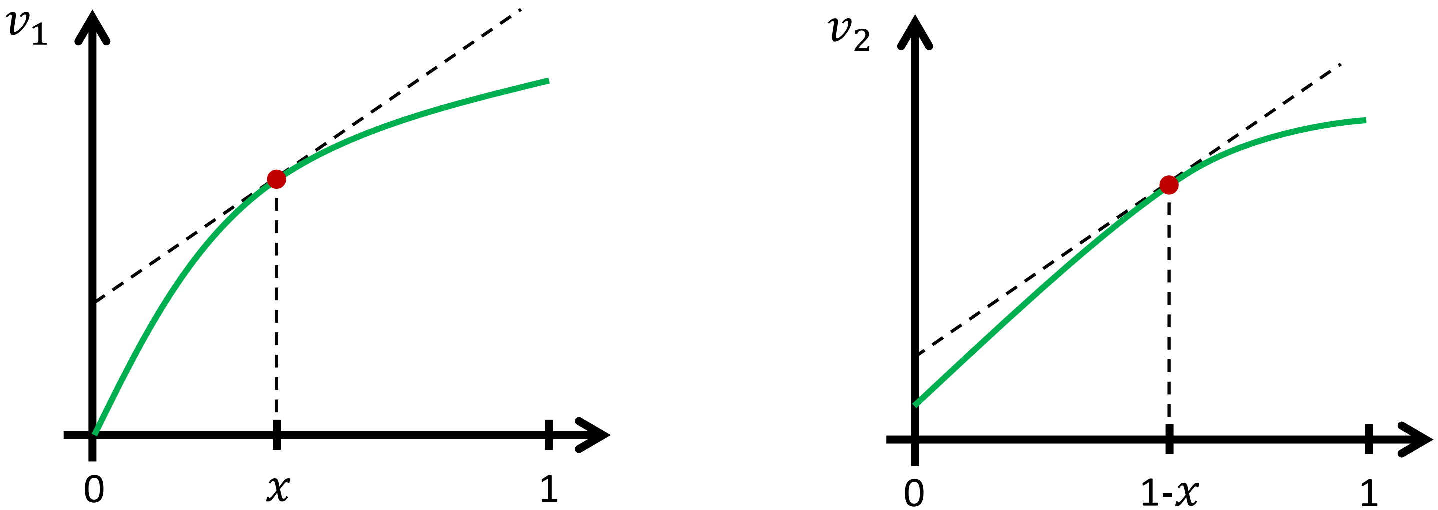
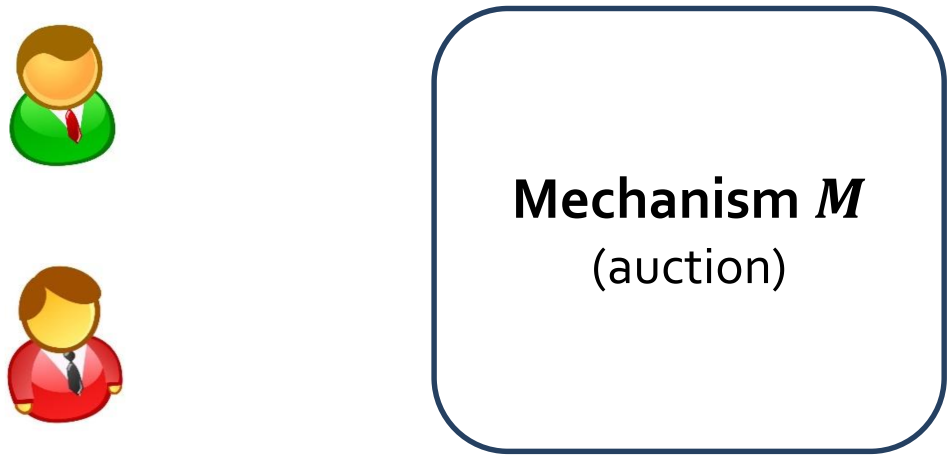


optimal allocation:

equal slopes

Resource allocation mechanisms



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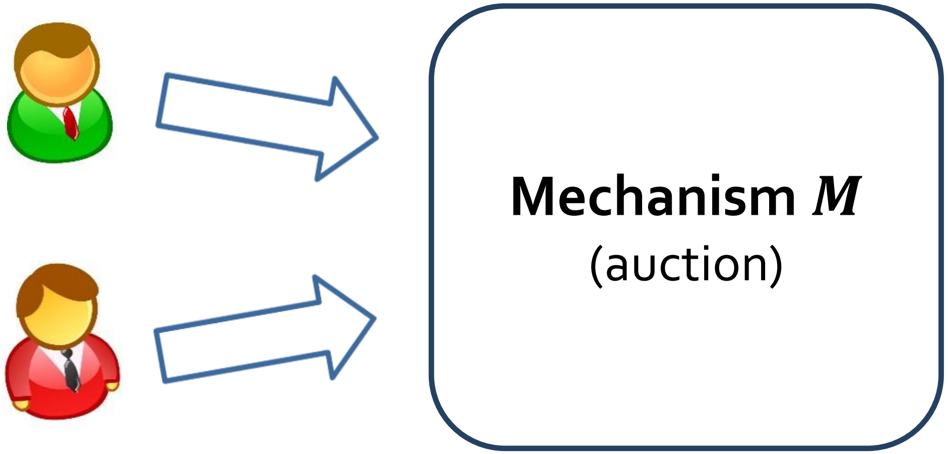
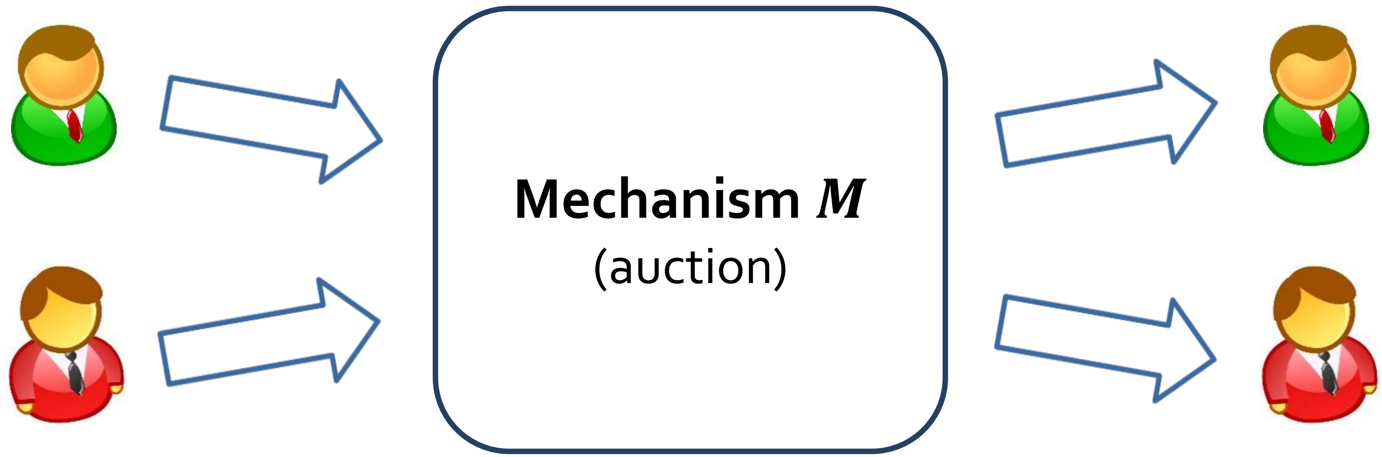


Input: signals (bids)

$$\boldsymbol{s} = (s_1, \ldots, s_n)$$

$$s_1, \ldots, s_n \geq 0$$

Resource allocation mechanisms



Input: signals (bids)

$$\boldsymbol{s} = (s_1, \ldots, s_n)$$

$$s_1, \ldots, s_n \geq 0$$

Output: allocation and payments

$$g(\boldsymbol{s}) = (g_1(\boldsymbol{s}), \ldots, g_n(\boldsymbol{s}))$$

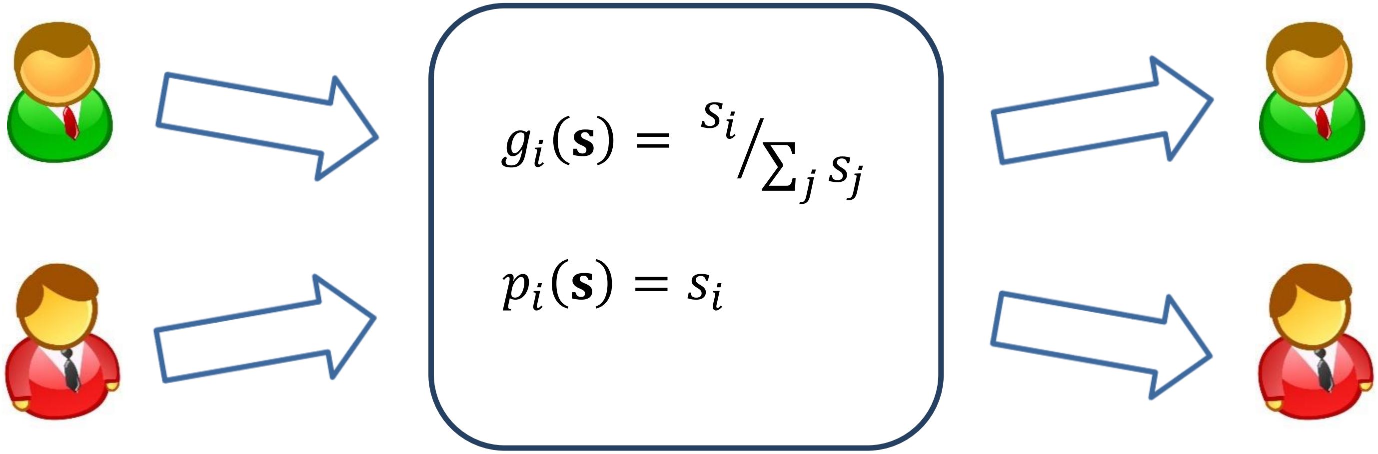
$$\sum_i g_i(\boldsymbol{s}) = 1$$

$$p(\boldsymbol{s}) = (p_1(\boldsymbol{s}), \ldots, p_n(\boldsymbol{s}))$$

$$p_1(\boldsymbol{s}), \ldots, p_n(\boldsymbol{s}) \geq 0$$

Some examples

- **Kelly mechanism (1997)**
  - proportional allocation + pay-your-signal



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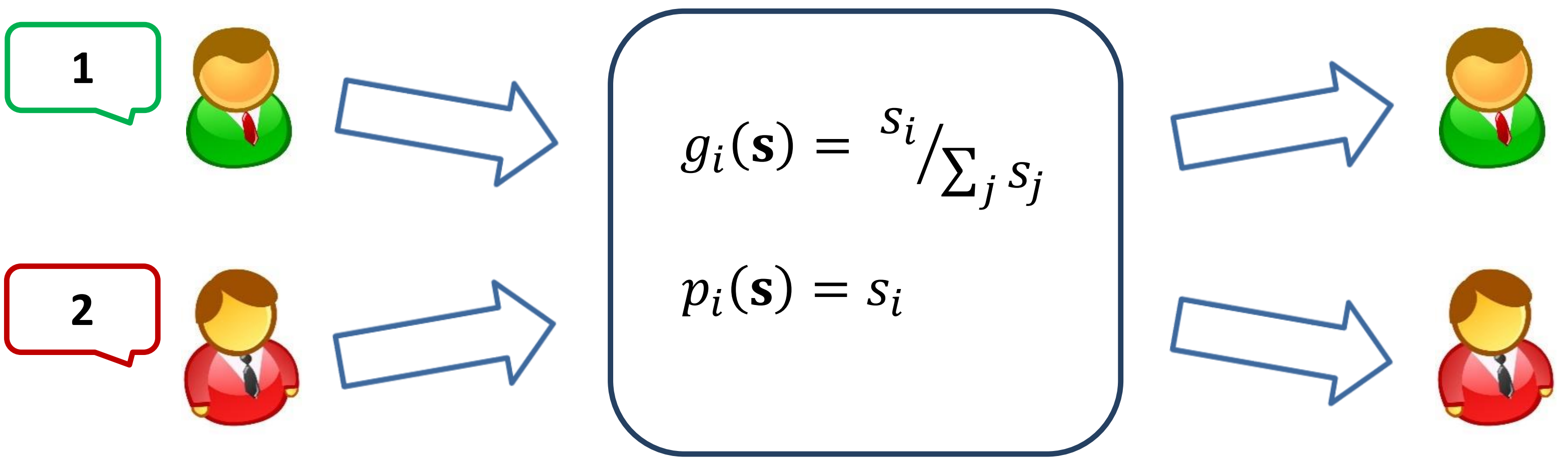

$$g_i(\mathbf{s}) = {}^{s_i}\!/_{\sum_j s_j}$$

$$p_i(\mathbf{s}) = s_i$$

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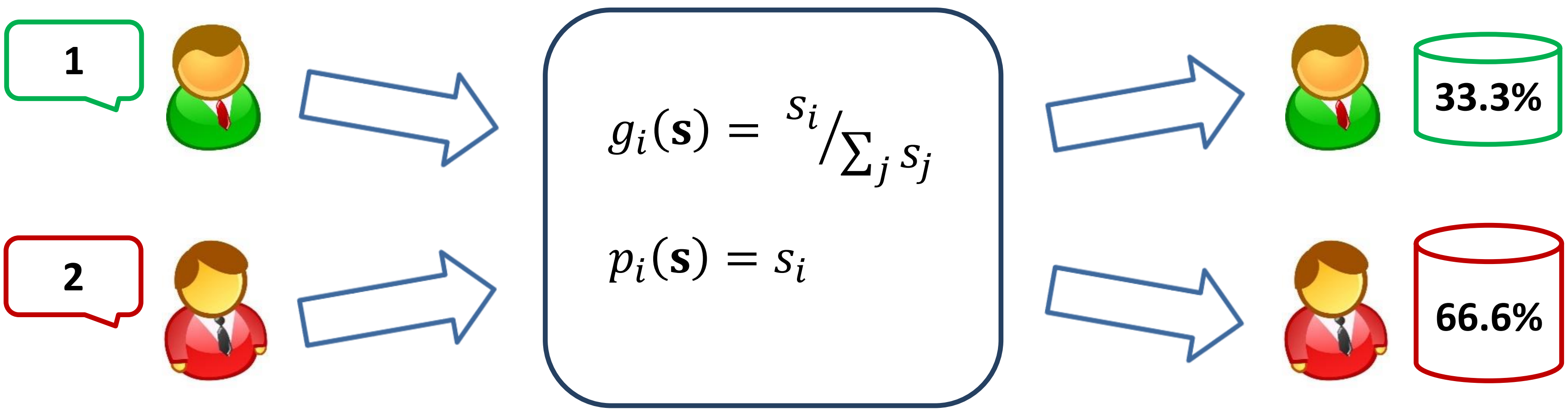
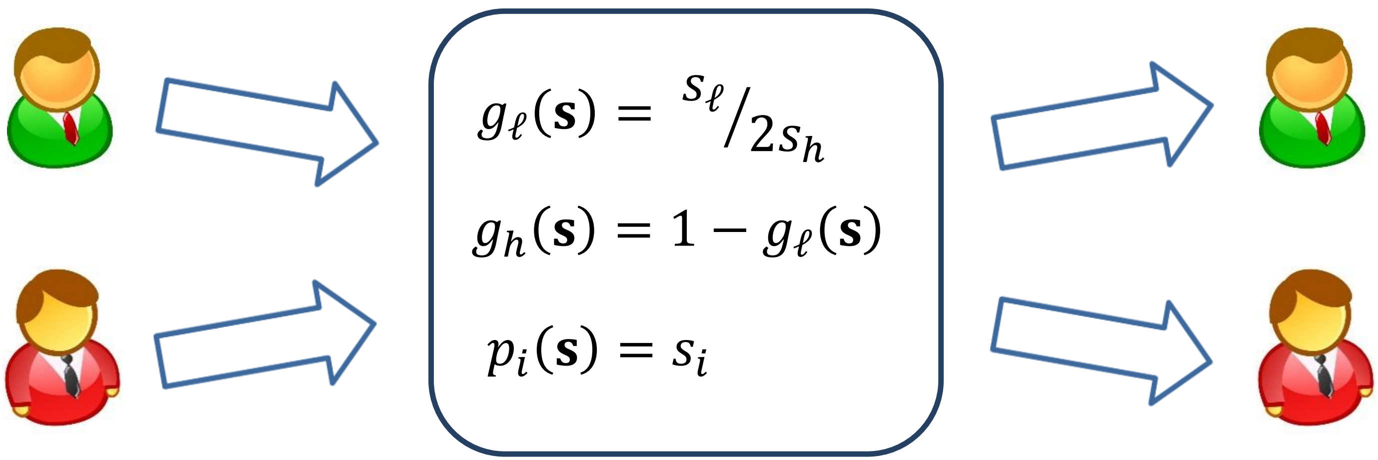


Some examples

- **Sanghavi and Hajek (SH) mechanism (2004)**
  - Allocation depending on highest signal + pay-your-signal



$$g_\ell(\mathbf{s}) = {}^{s_\ell}/_{2s_h}$$

$$g_h(\mathbf{s}) = 1 - g_\ell(\mathbf{s})$$

$$p_i(\mathbf{s}) = s_i$$

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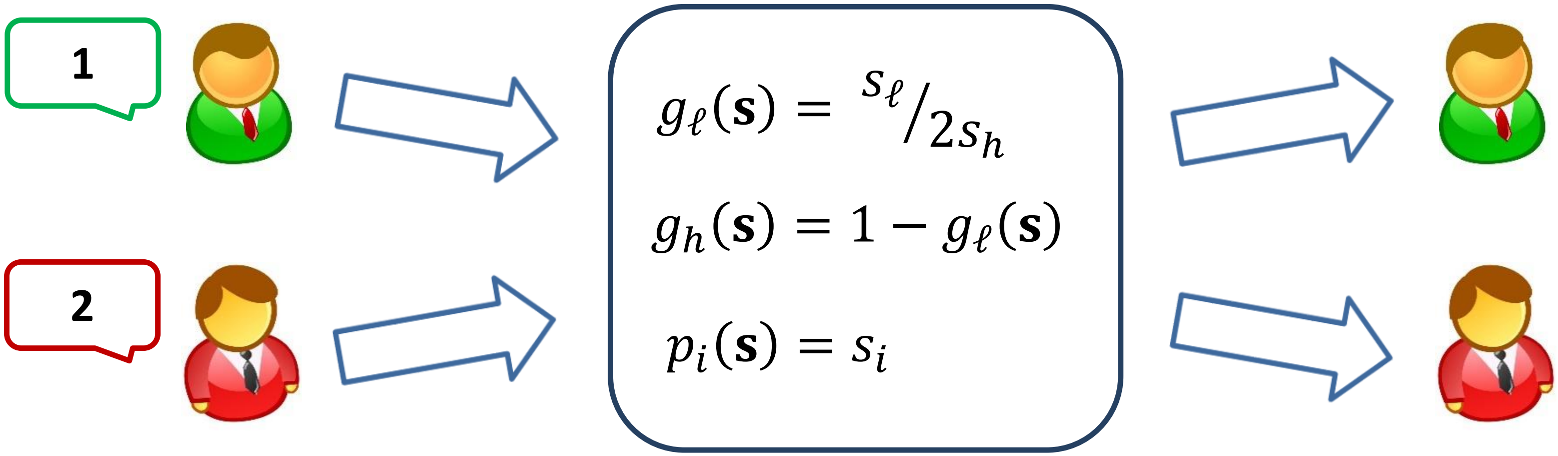

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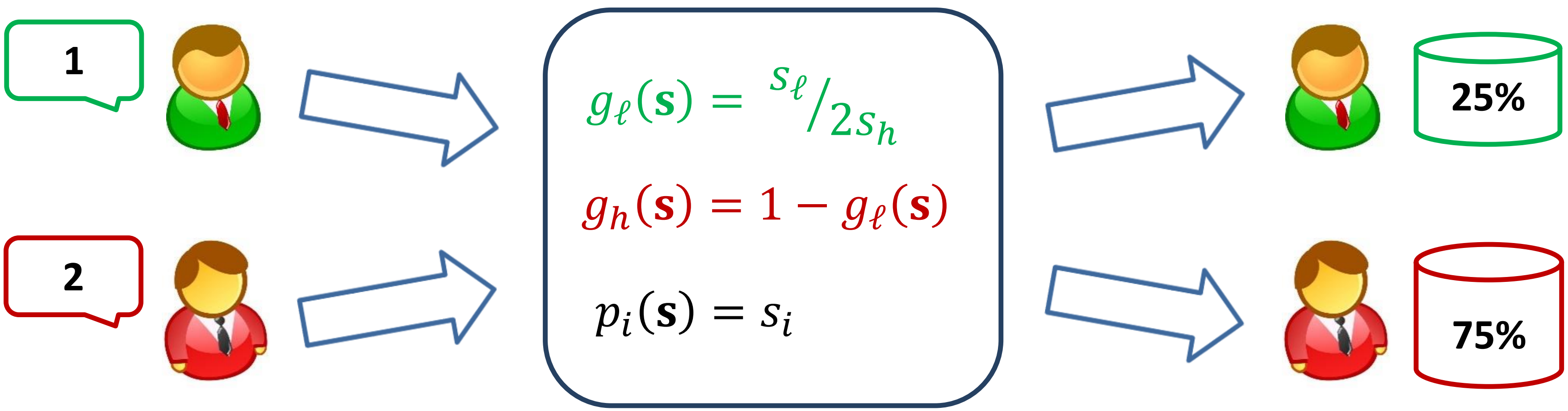

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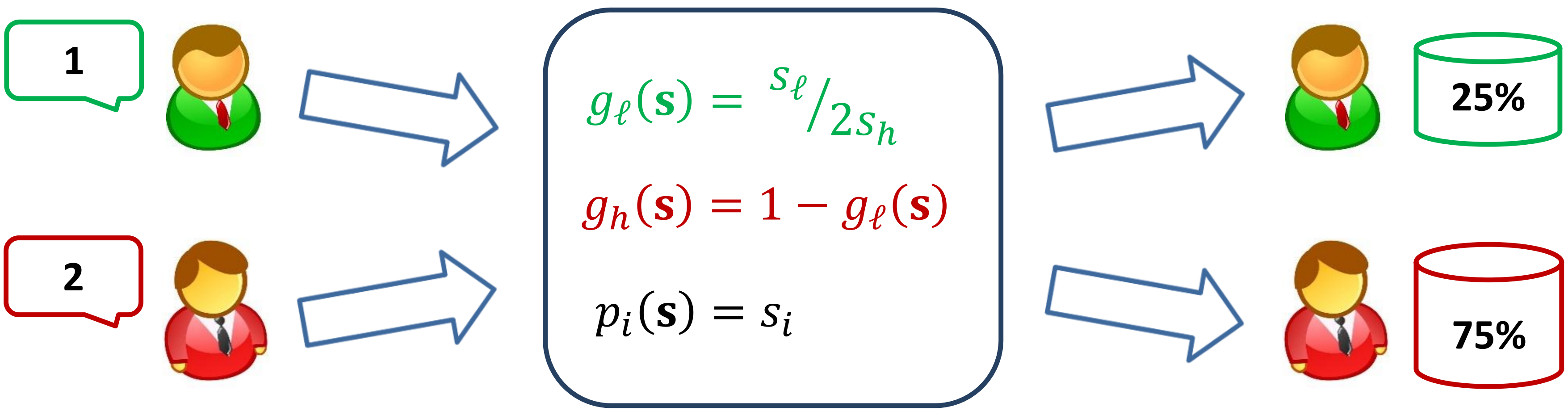
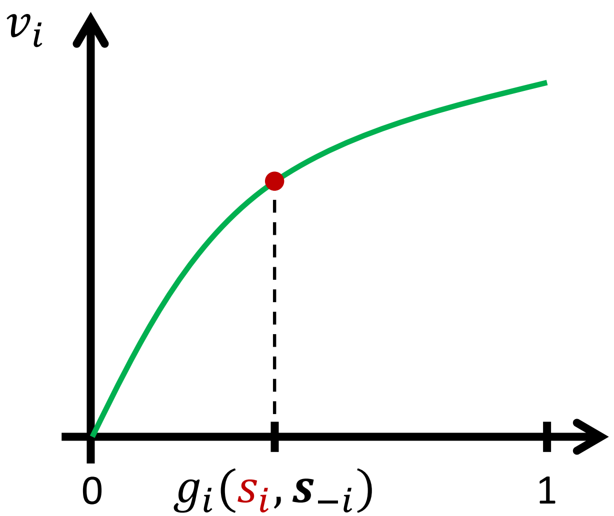


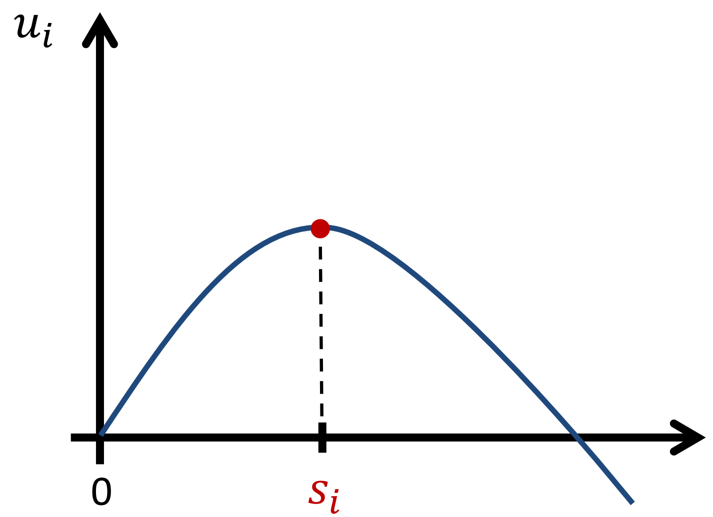
$$g_i(\mathbf{s}) = \frac{s_i}{\max_j s_j} \int_0^1 \prod_{k \neq i} \left(1 - \frac{s_k}{\max_j s_j} t\right) \mathrm{d}t$$

Strategic behavior

- Users are **utility-maximizers**

$$u_i(s_i, \boldsymbol{s}_{-i}) = \underbrace{v_i\big(g_i(s_i, \boldsymbol{s}_{-i})\big)}_{\text{value}} - \underbrace{p_i(s_i, \boldsymbol{s}_{-i})}_{\text{payment}}$$





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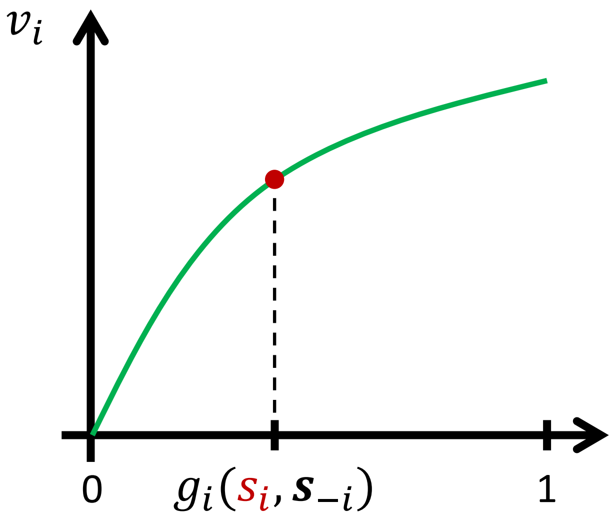
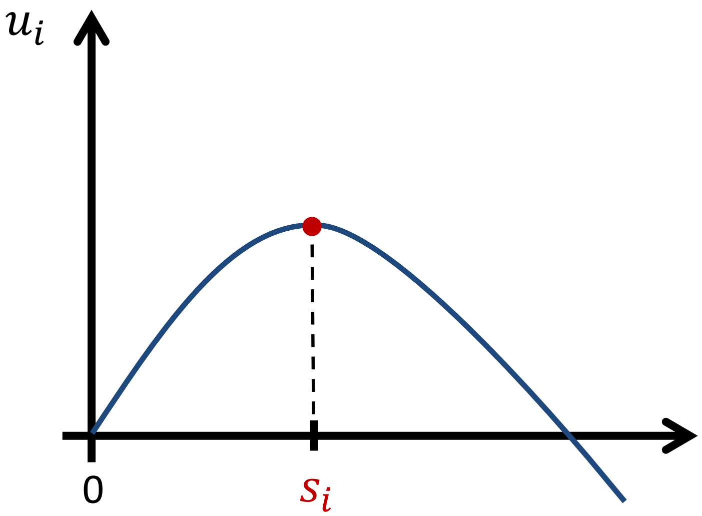
- **(Pure Nash) equilibrium:** Given the signals of the other users, all users submit signals that maximize their personal utilities
- **Efficiency** of mechanism $\boldsymbol{M}$: **price of anarchy** with respect to the social welfare

$$\text{PoA}(\boldsymbol{M}) = \sup_{\boldsymbol{v}} \frac{\max_{\boldsymbol{x}} \text{SW}(\boldsymbol{x})}{\min_{\boldsymbol{s}\in \text{EQ}(\boldsymbol{v},\boldsymbol{M})} \text{SW}(\boldsymbol{g}(\boldsymbol{s}))}$$

Worst-case characterization





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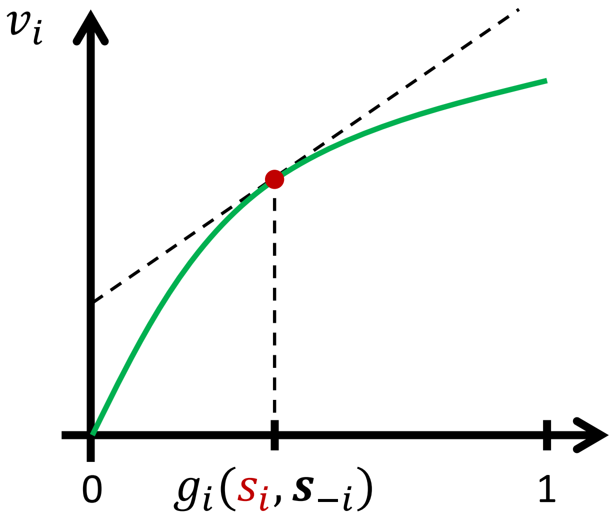

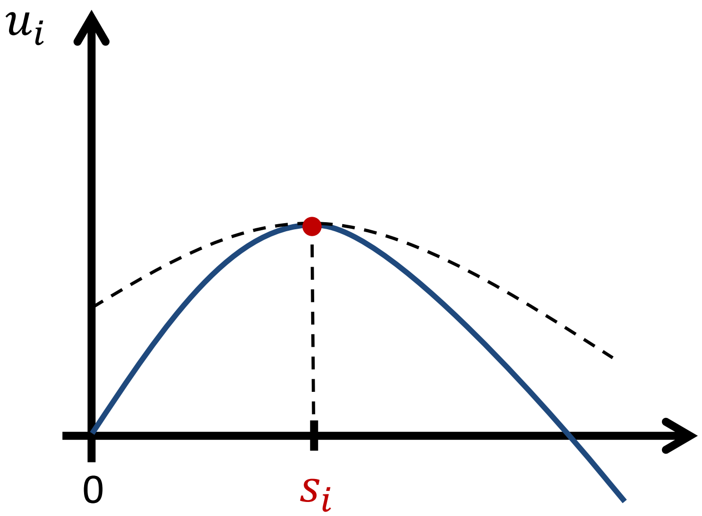

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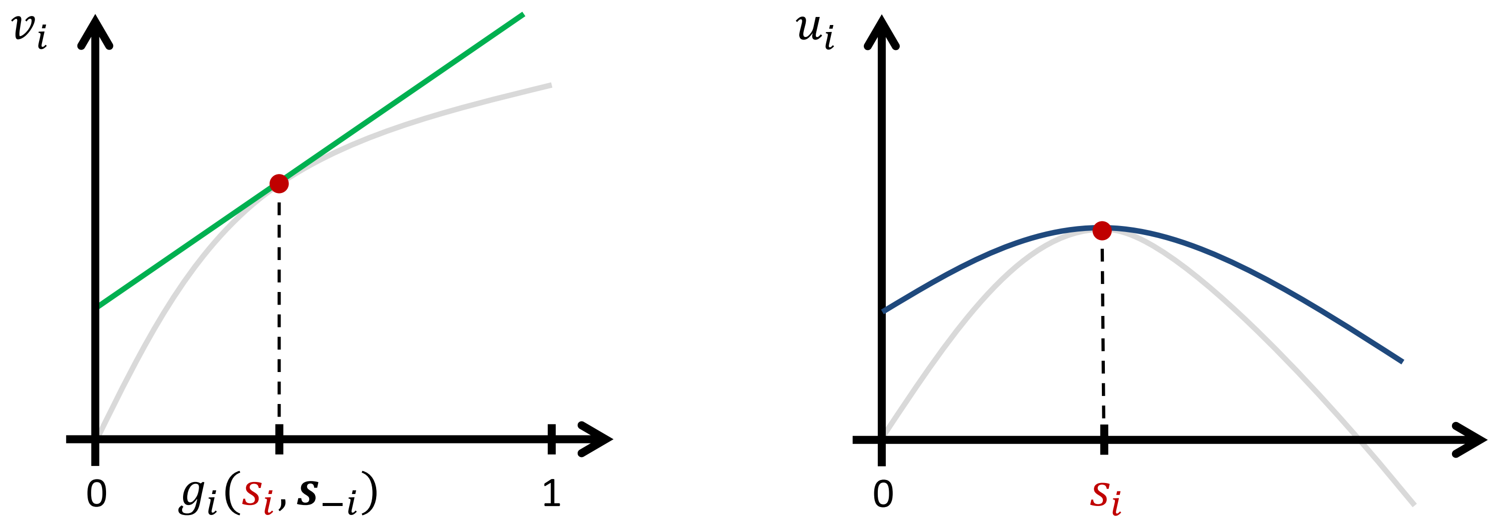

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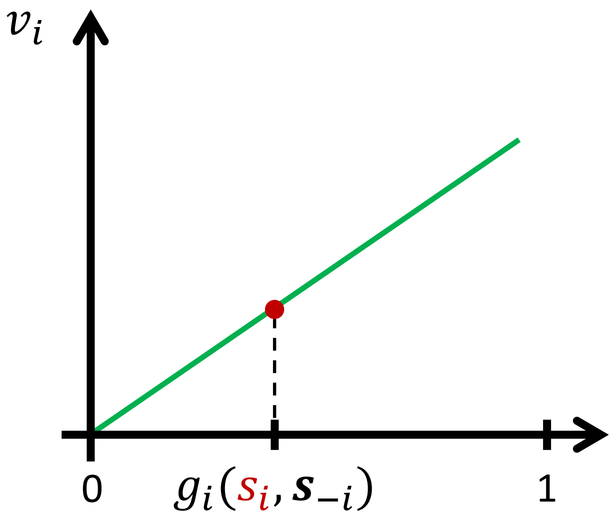

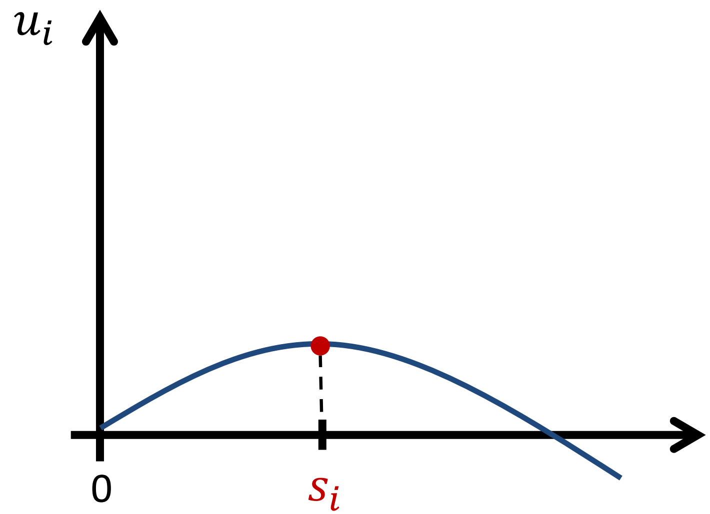

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- The **same** signal vector would still be an equilibrium if the valuation functions were linear
- The price of anarchy can only become **worse**

Known PoA bounds

- **PoA(Kelly) = 4/3**
  - Johari and Tsitsiklis (2004)
- **PoA(SH) ≈ 8/7**
  - Sanghavi and Hajek (2004)

- There exist mechanisms with **PoA = 1**
  - Maheswaran and Basar (2006)
  - Yang and Hajek (2007)
  - Johari and Tsitsiklis (2009)

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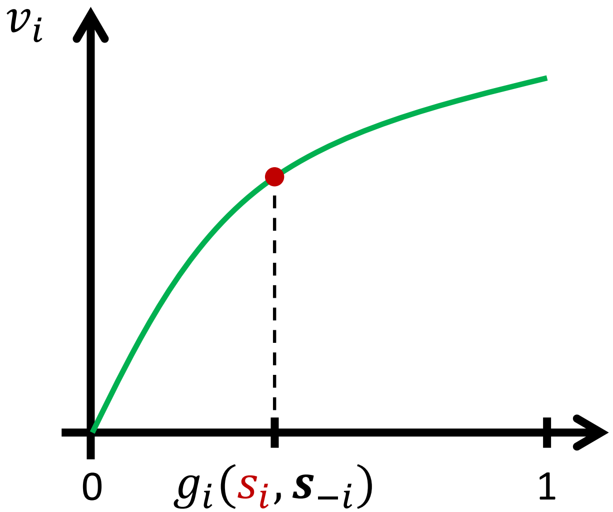

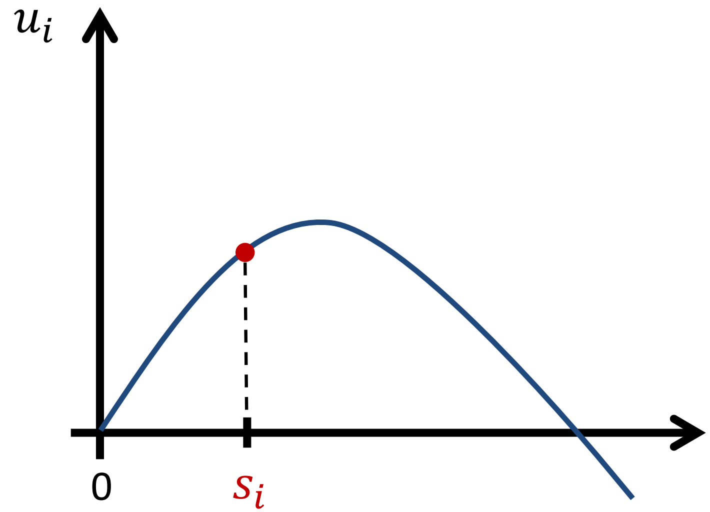

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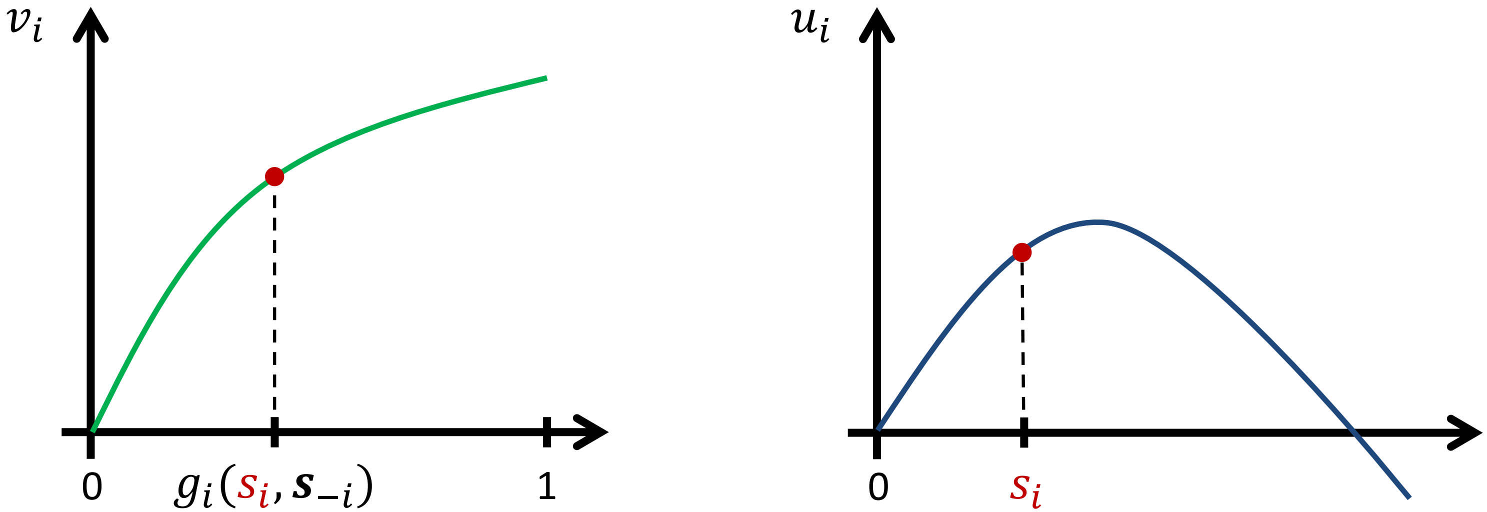

- **Different equilibria**

Efficiency under budget constraints

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- **Liquid welfare**

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  - Syrgkanis and Tardos (2013)
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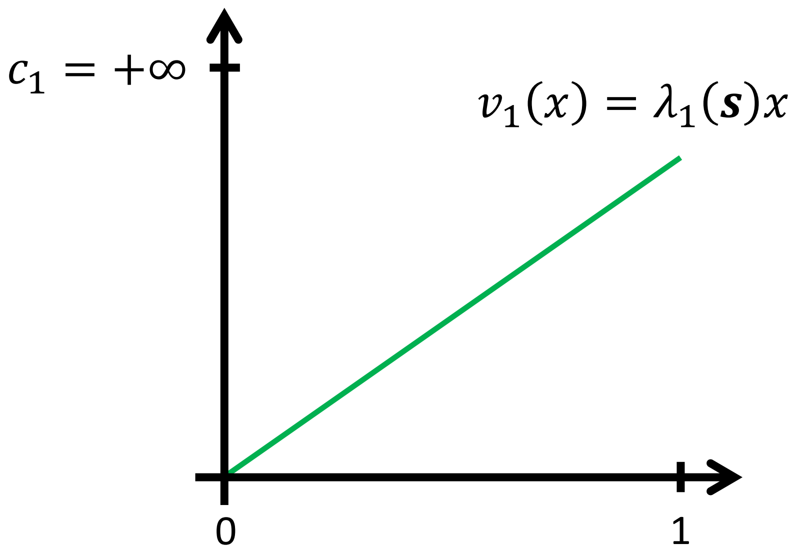

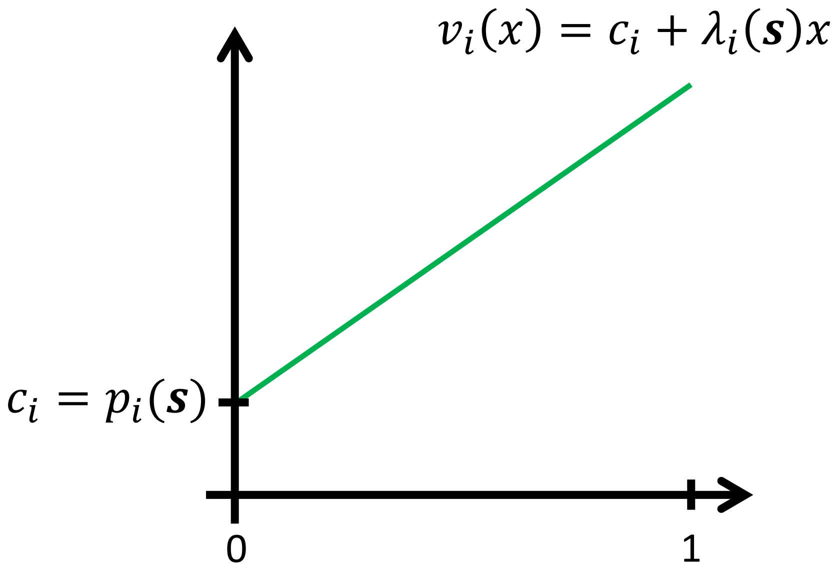

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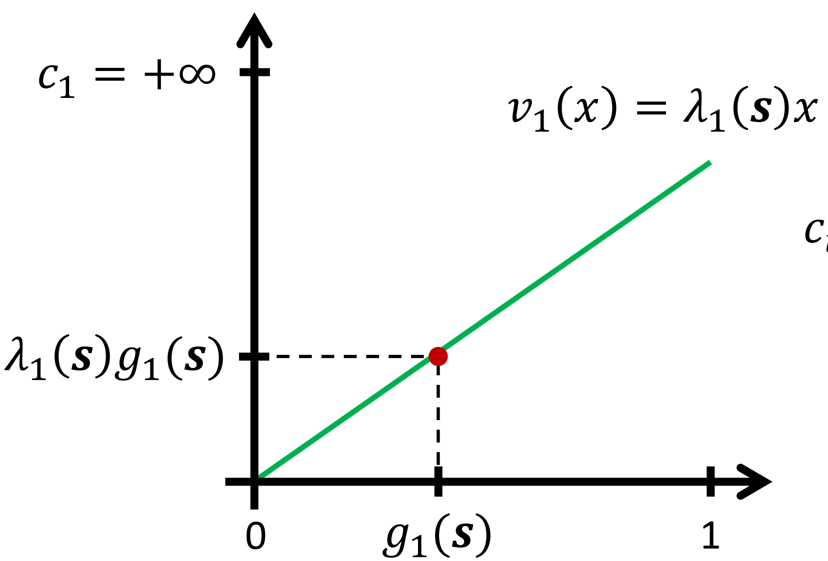

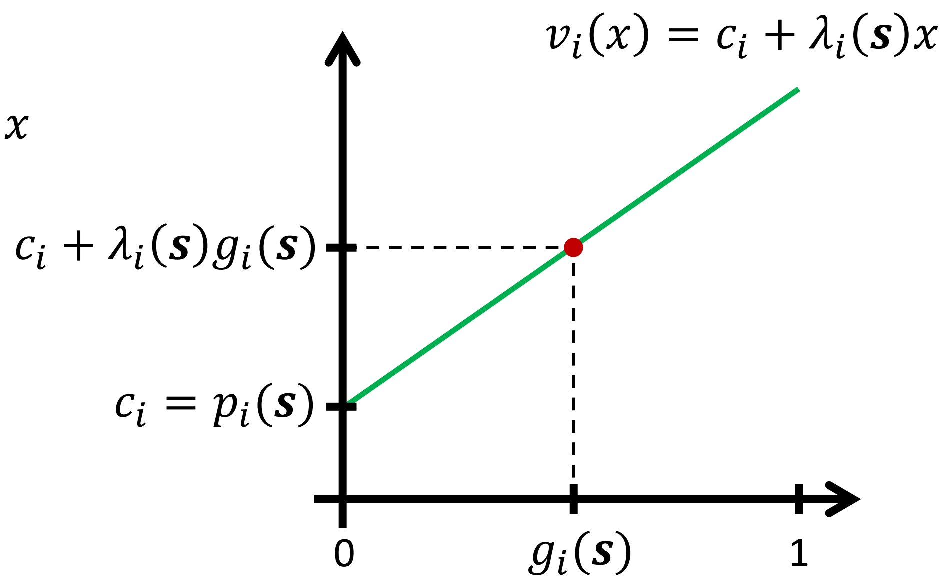

Equilibrium

$$\mathrm{LW}(g(\boldsymbol{s})) = \sum_{i\geq 2} p_i(s) + \lambda_1(\boldsymbol{s})g_1(\boldsymbol{s})$$

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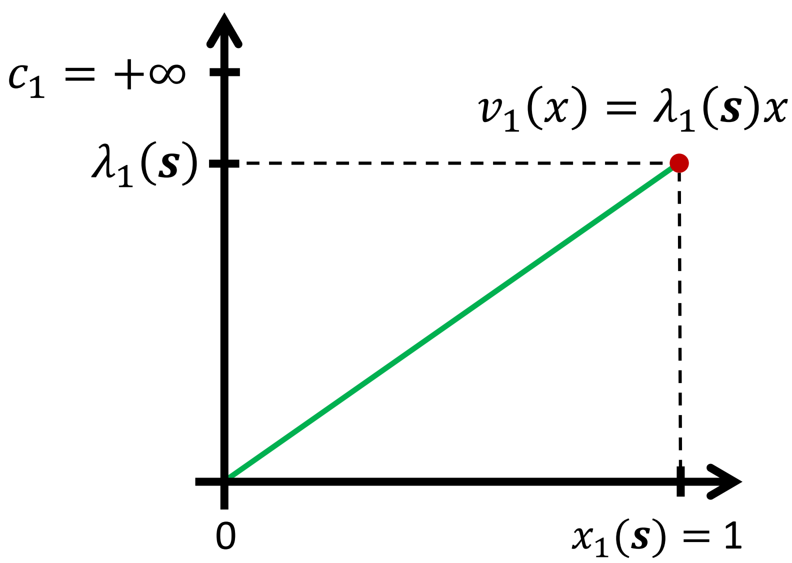

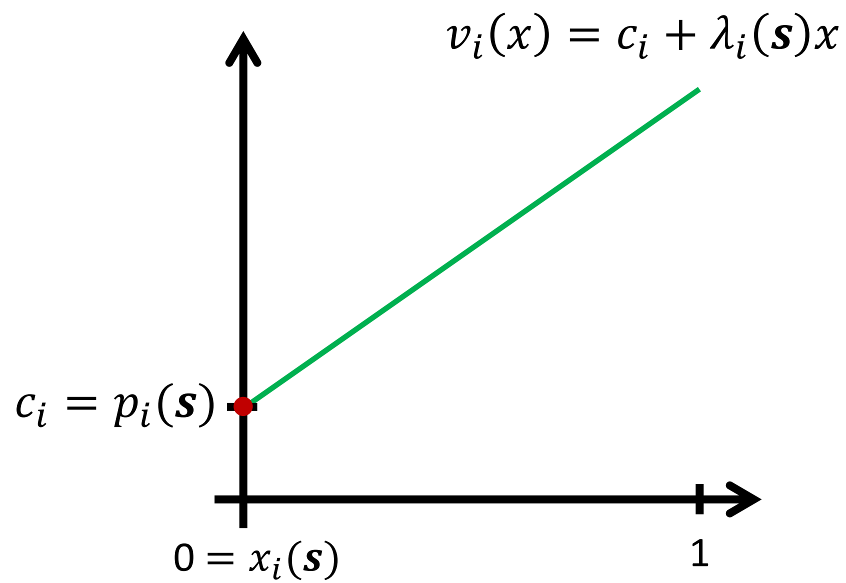

Optimal allocation

$$\mathrm{LW}(x(\boldsymbol{s})) = \sum_{i \geq 2} p_i(s) + \lambda_1(\boldsymbol{s})$$

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$$\text{LPoA}(\boldsymbol{s}\text{–game}) = \frac{\text{LW}(x(\boldsymbol{s}))}{\text{LW}(g(\boldsymbol{s}))} = \frac{\sum_{i\geq 2} p_i(\boldsymbol{s}) + \lambda_1(\boldsymbol{s})}{\sum_{i\geq 2} p_i(\boldsymbol{s}) + \lambda_1(\boldsymbol{s}) g_1(\boldsymbol{s})}$$

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- The LPoA of mechanism $\boldsymbol{M}$ is the worst LPoA over all $\boldsymbol{s}$-games

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Overview of results

mechanism	LPoA	
all	$\geq$ 2-1/n	No mechanism can achieve full efficiency
Kelly	2	Almost best possible among all mechanisms
SH	3	Different picture than the no-budget case
E2-PYS	1.79	Best possible 2-player PYS mechanism
E2-SR	1.53	almost best possible 2-player mechanism

Possible extensions

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- **Higher expressiveness:** the lower bound of 2-1/n still holds
- **Budget-aware mechanisms:** weaker lower bound of 4/3
- **More general equilibrium classes**

Thank you!