Bounding the inefficiency of compromise

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Paper overview

- We study a refinement of an opinion formation model proposed by Friedkin and Johnsen (1990), where each participant has an internal belief, but expresses a public opinion which might be affected by her social acquaintances
- We follow the recent game-theoretic approach of Bindel, Kleinberg, and Oren (2015) and focus on a co-evolutionary setting proposed by Bhawalkar, Gollapudi, and Munagala (2013), in which both the social acquaintances and the opinions co-evolve
- Specifically, we introduce a new cost function that better models the behavior of participants that aim to compromise, formulate corresponding strategic games, which we call k-COF (standing for "compromising opinion formation") games, and present results on the existence, complexity, and quality of their equilibria

k-COF games

- n players
- Vector $\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathbb{R}^n$ of player beliefs, such that $s_i \le s_{i+1}$ for each $i \in [n-1]$
- Vector $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n$ of opinions expressed by the players; these opinions define a state of the game
- Neighborhood $N_i(\mathbf{z}, \mathbf{s})$ of player i is the set of k players whose opinions are the closest to the belief of player i
- The cost of player i is

$$\text{cost}_i(\mathbf{z}, \mathbf{s}) = \max_{j \in N_i(\mathbf{z},\mathbf{s})} \left\{ |z_i - s_i|, |z_j - z_i| \right\}$$

- An opinion vector $\mathbf{z}$ is a *pure Nash equilibrium* if no player i has any incentive to unilaterally deviate to a deterministic opinion z'_i in order to decrease her cost, i.e.,

$$\text{cost}_i(\mathbf{z}, \mathbf{s}) \le \text{cost}_i((z'_i, \mathbf{z}_{-i}), \mathbf{s})$$

Structural properties

- For player i, define $I_i(\mathbf{z}, \mathbf{s})$ as the shortest interval of the real line that includes the belief s_i, the opinion z_i, and the opinion z_j for each player $j \in N_i(\mathbf{z}, \mathbf{s})$
- Also, define $\ell_i(\mathbf{z}, \mathbf{s})$ and $r_i(\mathbf{z}, \mathbf{s})$ as the players with the leftmost and rightmost point in $I_i(\mathbf{z}, \mathbf{s})$, respectively. For example, $\ell_i(\mathbf{z}, \mathbf{s})$ can be equal to either player i or some player $j \in N_i(\mathbf{z}, \mathbf{s})$, depending on whether the leftmost point of $I_i(\mathbf{z}, \mathbf{s})$ is s_i, z_i, or z_j

Lemma 1. *In any pure Nash equilibrium* $\mathbf{z}$ *of a* k*-COF game with belief vector* $\mathbf{s}$*,*

a. *z_i lies in the middle of interval $I_i(\mathbf{z}, \mathbf{s})$, for each player i;*

b. *$z_i \le z_{i+1}$, for any $i \in [n-1]$;*

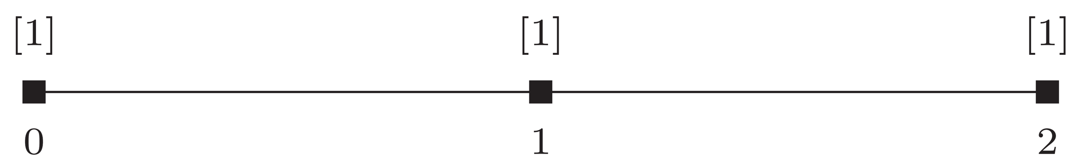
c. *$N_i(\mathbf{z}, \mathbf{s}) = \{j, ..., j+k\} \setminus \{i\}$ with $i - k \le j \le i$;*

d. *$s_{\ell_i(\mathbf{z},\mathbf{s})} \le z_i \le s_{r_i(\mathbf{z},\mathbf{s})}$, for each player i.*

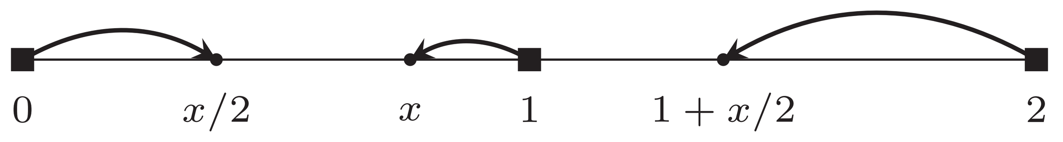
Non-existence of pure equilibria

Theorem 1. *There exist* 1*-COF games with no pure equilibria.*

In the examples below, squares are beliefs, points and arrows are opinions, and $[y]$ denotes y players with identical beliefs.



Let's assume, w.l.o.g., that the middle player plays opinion $x \le 1$. Then, by structural properties of pure equilibria, the other players have the middle player as neighbor and play at $x/2$ and $1 + x/2$, respectively.



If $x < 1$, the opinion that is closest to the belief of the middle player is that of the rightmost player and the middle player would have an incentive to deviate to the middle point $1 + x/4$ of the interval between her belief and the opinion of the rightmost player.

Now, let's assume that $x = 1$. Then, clearly, the middle player does not play in the middle of the interval between her belief and her neighbor's opinion.

Quantifying inefficiency: definitions

- Given belief vector $\mathbf{s}$, the *social cost* of opinion vector $\mathbf{z}$ is

$$\text{SC}(\mathbf{z}, \mathbf{s}) = \sum_i \text{cost}_i(\mathbf{z}, \mathbf{s})$$

- Given $k \ge 1$, the price of anarchy (PoA) is the worst-case ratio, over all belief vectors $\mathbf{s}$, between the maximum social cost at equilibrium and the optimal social cost:

$$\text{PoA} = \sup_{\mathbf{s} \in \mathbb{R}^n} \sup_{\mathbf{z} \in \text{PNE}(\mathbf{s})} \frac{\text{SC}(\mathbf{z}, \mathbf{s})}{\text{SC}(\mathbf{z}^*(\mathbf{s}), \mathbf{s})},$$

where $\mathbf{z}^*(\mathbf{s})$ is an optimal opinion vector and $\text{PNE}(\mathbf{s})$ denotes the set of pure Nash equilibria

- Similarly, the price of stability (PoS) is defined as:

$$\text{PoS} = \sup_{\mathbf{s} \in \mathbb{R}^n} \inf_{\mathbf{z} \in \text{PNE}(\mathbf{s})} \frac{\text{SC}(\mathbf{z}, \mathbf{s})}{\text{SC}(\mathbf{z}^*(\mathbf{s}), \mathbf{s})}$$

PoA/PoS bounds

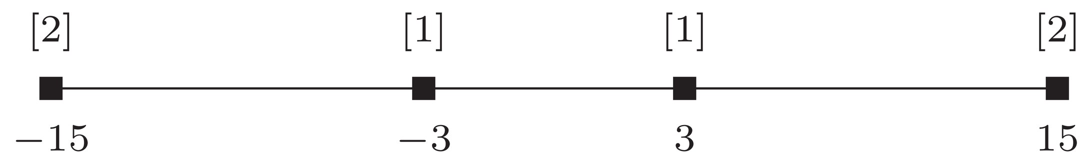
Theorem 2. *The price of anarchy of*

- 1*-COF games over pure equilibria is exactly* 3*;*
- k*-COF games over pure equilibria is at most* $4(k+1)$ *for any* $k \ge 2$*, at least* $18/5$ *for* $k = 2$*, and at least* $k+1$ *for* $k \ge 3$*;*
- k*-COF games over mixed equilibria is at least* 6 *for* $k = 1$*, at least* $24/5$ *for* $k = 2$*, and at least* $k+2$ *for* $k \ge 3$*.*

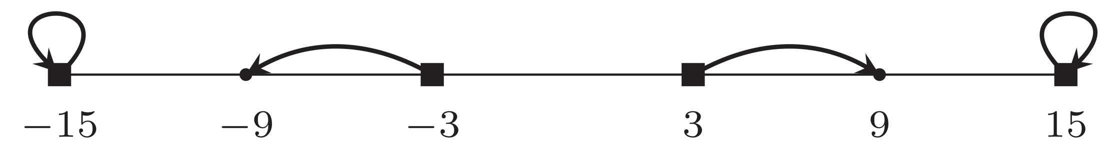
Theorem 3. *The price of stability of* k*-COF games is at least* $17/15$*.*

PoA lower bound for 1-COF games

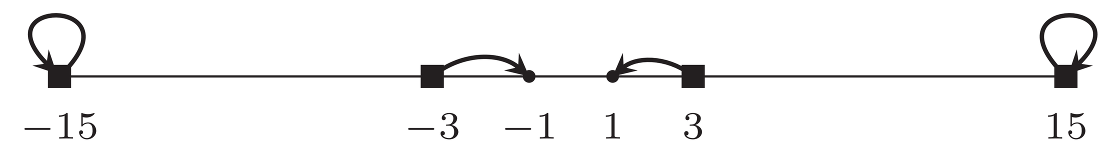
Consider the following 6-player 1-COF game:



A pure equilibrium with social cost 12. The two leftmost (resp., rightmost) players have each other as neighbor. The middle-left (resp., middle-right) player has a leftmost (resp., rightmost) player as neighbor.



The optimal state with social cost 4. Now, the two middle players have each other as neighbor.



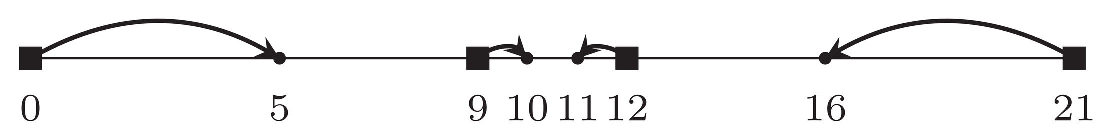
The price of anarchy is at least $12/4 = 3$.

Computing equilibria

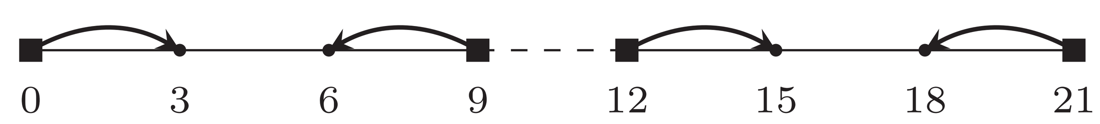
Theorem 4. *Deciding whether a* 1*-COF game has a pure equilibrium can be done in polynomial time. Furthermore, computing an equilibrium of highest or lowest social cost can be done in polynomial time as well.*

The algorithm: main ideas

The main idea is to decompose an equilibrium into legit segments. Consider, for example, four players with indices from 1 to 4 and belief vector $(0, 9, 12, 21)$. Here is how the legit segment $C(1, 2, 4)$ looks like:

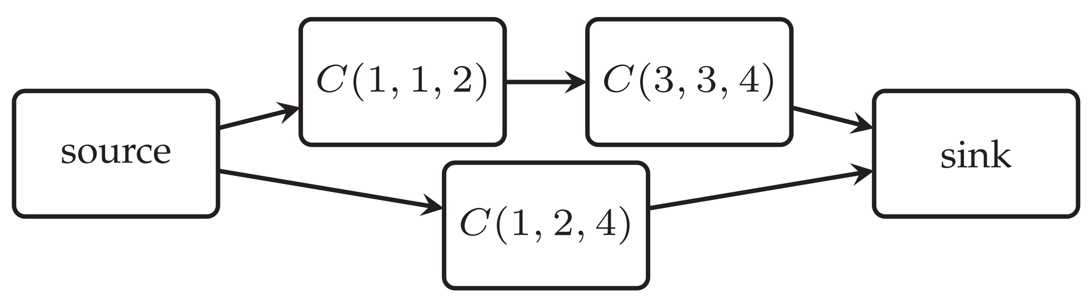


I.e., the first two players play to their right and the other ones play to their left. The next figure depicts the legit segments $C(1, 1, 2)$ and $C(3, 3, 4)$:



It can be seen that there are no other legit segments that have to be considered.

Now, the algorithm builds a directed graph showing how segments can be connected.



Source-sink paths correspond to equilibria.

More info

I. Caragiannis, P. Kanellopoulos, and A. A. Voudouris. Bounding the inefficiency of compromise. *arXiv: 1702.07309*