Efficiency and Complexity of Price Competition Among Single-Product Vendors

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Model

- There is a set M of m vendors each selling a single product; vendor j has production cost c_j.
- The objective of each vendor is to determine a price $p_j \geq c_j$ for its product; $\mathbf{p}$ denotes a price vector containing a price per vendor.
- There is a large volume of unit-demand buyers classified into a set N of n distinct types; buyer type i has volume μ_i and valuation v_{ij} for the product of vendor j.
- After prices are set, the buyers select which products they will buy; $D_i(\mathbf{p})$ is the set of vendors whose prices maximize the utility of buyers of type i.
- A buyers-to-vendors *assignment* $\mathbf{x}$ denotes how the volume of the buyers of each type is split among the vendors; $\mathbf{x}$ is *consistent* to a price vector $\mathbf{p}$ if $x_{ij} > 0$ implies $j \in D_i(\mathbf{p})$.
- The *social welfare* of an assignment is

$$\mathrm{SW}(\mathbf{x}) = \sum_{i\in N}\sum_{j\in M} x_{ij}(v_{ij} - c_j).$$

- The *optimal social welfare* is defined as

$$\mathrm{SW}^* = \sum_{i\in N} \mu_i \max_{j\in M}\{v_{ij} - c_j\}$$

Price Competition Game

- *Two-stage game*: in the first stage, the vendors set a price for their products; in the second stage, the buyers select from whom to buy (or abstain).
- A price vector $\mathbf{p}$ and a consistent assignment $\mathbf{x}$ form a (pure Nash) equilibrium if the utility of every vendor j, denoted by $u_j(\mathbf{x}, (p'_j, \mathbf{p}_{-j}))$, is maximized among all prices $p'_j \geq c_j$ and all assignments $\mathbf{y}$ that are consistent to $(p'_j, \mathbf{p}_{-j})$).
- The quality of equilibria is measured by the price of anarchy, defined as

$$\mathrm{PoA} = \frac{\mathrm{SW}^*}{\min_{(\mathbf{x},\mathbf{p})\in \mathrm{PNE}} \mathrm{SW}(\mathbf{x})}.$$

Example

- Consider two software companies each developing an operating system with production costs 0.
- There are two buyer types with volumes $\mu_1 = \mu_2 = 1$, and valuations $v_1 = (6, 2)$ and $v_2 = (1, 5)$.
- The price vector $\mathbf{p} = (6, 5)$ and the consistent assignment $\mathbf{x}$ with $x_{11} = 1$ and $x_{22} = 1$, can be verified to be an equilibrium.
- For example, if vendor 1 deviates to a price so that it attracts both types of buyers, then this price will be at most 1 for a maximum utility of 2; this is less than its current utility 6 and, so, this vendor has no incentive to deviate to such a price.

Questions

- Do equilibria exist?
- What is their quality?
- Can we compute them efficiently?
- Can we enforce the optimal assignment as an equilibrium?

Existence of Equilibria

- Price competition games with one buyer type always have at least one equilibrium.
- There exists a price competition game with two buyer types that admits no equilibrium.

Price of Anarchy

- The price of anarchy of any price competition game with n buyer types is at most n.
- There are one-vendor price competition games with price of anarchy that is arbitrarily close to n.

Computational Problems

VerifyEquilibrium: Given a price vector $\mathbf{p}$ and a buyers-to-vendors assignment $\mathbf{x}$ in a price competition game, decide whether $(\mathbf{x}, \mathbf{p})$ is an equilibrium.

- Solvable in time $\mathcal{O}(nm)$.

ComputePrice: Given a buyers-to-vendors assignment $\mathbf{x}$, decide whether there exists a price vector $\mathbf{p}$ to which $\mathbf{x}$ is consistent so that $(\mathbf{x}, \mathbf{p})$ is an equilibrium.

- Efficiently solvable using the poly-time `CandidatePrice` algorithm.

PriceCompetition: Given a price competition game, decide whether it has any equilibrium or not.

- Efficiently solved if number of buyer types or number of vendors is constant.
- NP-hard in general: reduction from Exact-3-Cover.

`CandidatePrice`

- Given as input a buyers-to-vendors assignment $\mathbf{x}$, it returns a price vector $\mathbf{p}$; if $(\mathbf{x}, \mathbf{p})$ is not an equilibrium, then no equilibrium exists.
- It works as follows:
  - It computes a set Z of *seed vendors*; every such vendor j has price $p_j = c_j$.
  - If $Z = \emptyset$, then the price p_j of every other vendor j is

$$p_j = \min_{i:x_{ij}>0} v_{ij}.$$

  Otherwise, it is

$$p_j = \min_{i:x_{ij}>0}\left\{v_{ij} - \max_{j'\in Z}\{v_{ij'} - c_{j'}\}^+\right\}.$$

Introducing Subsidies

- Use external payments to vendors in order to incentivize them to lower their prices and enforce more efficient buyer-to-vendor assignments as equilibria.
- For a given price vector $\mathbf{p}$ and a consistent assignment $\mathbf{x}$, let $\theta_j(\mathbf{x}, \mathbf{p})$ denote the maximum possible utility of vendor j (over all deviating prices).
- In order to enforce $(\mathbf{x}, \mathbf{p})$ as an equilibrium, we need to pay an amount of

$$s_j(\mathbf{x}, \mathbf{p}) = \theta_j(\mathbf{x}, \mathbf{p}) - u_j(\mathbf{x}, \mathbf{p})$$

  to every vendor j so that j does not have an incentive to deviate to a price different than p_j.

Subsidies: Results

MinSubsidies: Given a price competition game with an optimal assignment $\mathbf{x}$, compute a price vector $\mathbf{p}$ to which $\mathbf{x}$ is consistent that minimizes the entry-wise subsidy vector $\mathbf{s}(\mathbf{x}, \mathbf{p})$ necessary to enforce $(\mathbf{x}, \mathbf{p})$ as an equilibrium.

- NP-hard to approximate within any constant: approximation-preserving reduction from NodeCover on k-uniform hypergraphs.
- In every price competition game, the optimal assignment can be enforced as an equilibrium using an amount of subsidies that is at most SW^*. This bound is tight.
- For every $\delta > 0$, there exists a price competition game, in which no subsidy assignment of total amount smaller than $(1/4 - \delta)\mathrm{SW}^*$ can enforce any price vector/consistent buyers-to-vendors assignment as an equilibrium.

Open Problems

- Are there FPT algorithms for PriceCompetition with respect to different parameters?
- Are there (for example) logarithmic approximation algorithms for MinSubsidies?
- What happens when the vendors have limited supply?
- Imperfect information setting and generalized equilibrium concepts.

More Info

I. Caragiannis, X. Chatzigeorgiou, P. Kanellopoulos, G. A. Krimpas, N. Protopapas, and A. A. Voudouris. Efficiency and complexity of price competition among single-product vendors. In *Proceedings of the 24th International Joint Conference on Artificial Intelligence (IJCAI)*, 2015.