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IMPROVED FPTAS FOR MULTIOBJECTIVE
SHORTEST PATHS WITH APPLICATIONS

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Improved FPTAS for Multiobjective Shortest Paths with Applications*

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Abstract

We consider multiobjective shortest paths, a fundamental (NP-hard) problem in multiobjective optimization, where we are interested not in optimizing a single objective, but in finding a set of paths that captures the trade-off (the so-called *Pareto curve*) among several objectives in a digraph whose edges are associated with multidimensional cost vectors. We provide a new FPTAS for computing an approximate Pareto curve for multiobjective shortest paths that significantly improves upon previous approaches, especially in the case of more than two objective functions. We show how our new FPTAS can be used to provide better approximate solutions to three related problems (multiobjective constrained optimal paths, multiobjective constrained paths, and non-additive shortest paths) that have important applications in the areas of quality of service routing in communication networks, and in computing traffic equilibria in transportation networks. We also show how our new FPTAS for multiobjective shortest paths can provide efficient approximate solutions in a completely different context – to a natural generalization of the weighted multicommodity flow problem with elastic demands and values that models several realistic design scenarios in transportation and communication networks. Finally, as a byproduct of our framework, we provide a generic method for constructing FPTAS for *any* multiobjective optimization problem with non-linear objective functions of a rather general form.

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1 Introduction

Multiobjective shortest paths (MOSP) is a core problem in the area of multiobjective optimization (an area under intense study within Operations Research and Economics in the last 60 years [6, 7]) with numerous applications that span from traffic optimization to quality-of-service (QoS) routing in networks and multicriteria decision making [9]. Informally, the problem consists in finding a set of paths that captures not a single optimum but the trade-off among $d > 1$ objective functions in a digraph whose edges are associated with d-dimensional attribute (cost) vectors.

In general, an instance of a multiobjective optimization problem is associated with a set of feasible solutions Q and a d-vector function $\mathbf{f} = [f_1, \ldots, f_d]^T$ (d is typically a constant) associating each feasible solution $q \in Q$ with a d-vector $\mathbf{f}(q)$ (w.l.o.g we assume that all objectives f_i, $1 \leq i \leq d$, are to be minimized). In a multiobjective optimization problem, we are interested not in finding a single optimal solution, but in computing the trade-off among the different objective functions, called the *Pareto curve* $\mathcal{P}$, which is the set of all feasible solutions in Q whose vector of the various objectives is *not* dominated by any other solution (a solution p *dominates* another solution q iff $f_i(p) \leq f_i(q)$, $\forall 1 \leq i \leq d$). Multiobjective optimization problems are usually NP-hard (as indeed is the case for MOSP). This is due to the fact that the Pareto curve is typically exponential in size (even in the case of two objectives). On the other hand, even if a decision maker is armed with the entire Pareto curve, s/he is left with the problem of which is the "best" solution for the application at hand. Consequently, three natural approaches to solve multiobjective optimization problems are to: (i) study approximate versions of the Pareto curve; (ii) optimize one objective while bounding the rest (*constrained approach*); and (iii) proceed in a normative way and choose the "best" solution by introducing a utility (typically non-linear) function on the objectives (*normalization approach*). In this paper, we investigate all these approaches for the multiobjective shortest path problem. In particular, we provide an improved FPTAS for MOSP and show how this can be used to obtain efficient approximate solutions to the *multiple constrained (optimal) path* problems (constrained approach), and to the *non-additive shortest path* problem (normalization approach). We also show how efficient approximate solutions can be obtained in a completely different context; namely, to a natural generalization of the weighted multicommodity flow problem with elastic demands and values that models several realistic scenarios in transportation and communication networks.

1.1 Multiobjective Shortest Paths

Despite so much research in multiobjective optimization [6, 7], only recently a systematic study of the complexity issues regarding the construction of approximate Pareto curves has been initiated [23, 28]. Informally, an $(1+\varepsilon)$-*Pareto curve* $\mathcal{P}_\varepsilon$ is a subset of feasible solutions such that for any Pareto optimal solution, there exists a solution in $\mathcal{P}_\varepsilon$ that is no more than $(1+\varepsilon)$ away in all objectives. Papadimitriou and Yannakakis in a seminal work [23] show that for any multiobjective optimization problem there exists a $(1+\varepsilon)$-Pareto curve $\mathcal{P}_\varepsilon$ of (polynomial) size $|\mathcal{P}_\varepsilon| = O((4B/\varepsilon)^{d-1})$, where B is the number of bits required to represent the values in the objective functions (bounded by some polynomial in the size of the input), that can be constructed by $O((4B/\varepsilon)^d)$ calls to a GAP routine that solves (in time polynomial in the size of the input and $1/\varepsilon$) the following problem: given a vector of values $\mathbf{a}$, either compute a solution that dominates $\mathbf{a}$, or report that there is no solution better than $\mathbf{a}$ by at least a factor of $1+\varepsilon$ in all objectives. Extensions to that method to produce a constant approximation to the smallest possible $(1+\varepsilon)$-Pareto curve for the cases of 2 and 3 objectives are presented in [28], while for $d > 3$ objectives inapproximability results are shown for such a constant approximation.

For the case of MOSP (and some other problems with linear objectives), Papadimitriou and Yannakakis [23] show how a GAP routine can be constructed (based on a pseudopolynomial algorithm for computing exact paths), and consequently provide a FPTAS for this problem. Note

		Best previous	This work
General digraphs	$d = 2$	$O\left(nm\frac{1}{\varepsilon}\log\left(nC^{max}\right)\left(\log\log n + \frac{1}{\varepsilon}\right)\right)$ [28]	$O\left(n^2m\frac{1}{\varepsilon}\log\left(nC^{max}\right)\right)$
	$d > 2$	$O((\log(nC^{max})/\varepsilon)^d \cdot T_{GAP})$ [23]	$O\left(nm\left(\frac{n\log(nC^{max})}{\varepsilon}\right)^{d-1}\right)$
DAGs	$d = 2$	$O\left(nm\frac{1}{\varepsilon}\log n\log\left(nC^{max}\right)\right)$ [30] $O\left(nm\frac{1}{\varepsilon^2}\log\left(nC^{max}\right)\right)$ [28]	$O\left(nm\frac{1}{\varepsilon}\log\left(nC^{max}\right)\right)$
	$d > 2$	$O\left(nm(\frac{n\log(nC^{max})}{\varepsilon})^{d-1}\log^{d-2}(\frac{n}{\varepsilon})\right)$ [30]	$O\left(m\left(\frac{n\log(nC^{max})}{\varepsilon}\right)^{d-1}\right)$

Table 1: Comparison of new and previous results for MOSP. T_{GAP} denotes the time for a call to a GAP routine, which is polynomial in the input and $1/\varepsilon$ (but exponential in d).

that FPTAS for MOSP were already known in the case of two objectives [17], as well as in the case of multiple objectives in directed acyclic graphs (DAGs) [30]. In particular, the 2-objective case has been extensively studied [7], while for $d > 2$ very little has been achieved; actually (to the best of our knowledge) the results in [23, 30] are the only and currently best FPTAS known. Let C^{max} denote the ratio of the maximum to the minimum edge weight (in any dimension), and let n (resp. m) be the number of nodes (resp. edges) in a digraph. For the case of DAGs and $d > 2$, the algorithm of [30] runs in $O(nm(\frac{n\log(nC^{max})}{\varepsilon})^{d-1}\log^{d-2}(\frac{n}{\varepsilon}))$ time, while for $d = 2$ this improves to $O(nm\frac{1}{\varepsilon}\log n\log(nC^{max}))$. For $d = 2$, a FPTAS can be created by repeated applications of a stronger variant of the GAP routine – like a FPTAS for the restricted shortest path (RSP) problem [8, 18, 21]. In [28] it is shown that this achieves a time of $O(nm|\mathcal{P}^*_\varepsilon|(\log\log n + 1/\varepsilon))$ for general digraphs and $O(nm|\mathcal{P}^*_\varepsilon|/\varepsilon)$ for DAGs, where $|\mathcal{P}^*_\varepsilon|$ is the size of the smallest possible $(1+\varepsilon)$-Pareto curve (and which can be as large as $\log_{1+\varepsilon} nC^{max} \approx \frac{1}{\varepsilon}\ln(nC^{max})$). All these approaches deal typically with the single-pair version of the problem.

Our main contribution in this work (Section 3) is a new and remarkably simple FPTAS for constructing a set of approximate Pareto curves for the *single-source* version of the MOSP problem in *any* digraph. For any $d > 1$, our algorithm runs in time $O(nm(\frac{n\log(nC^{max})}{\varepsilon})^{d-1})$ for general digraphs, and in $O(m(\frac{n\log(nC^{max})}{\varepsilon})^{d-1})$ for DAGs. These results improve significantly upon previous approaches for general digraphs [23, 28] and DAGs [28, 30], for all $d > 2$. For $d = 2$, our running times depend on ε^{-1}, while those based on repeated-RSP applications (like in [28]) depend on ε^{-2}. This always gives us better running times for DAGs, while for general digraphs we improve the dependency on $\frac{1}{\varepsilon}$. Table 1 summarizes the comparison of our results with the best previous ones.

Our approach for MOSP, unlike previous methods that are based on converting pseudopolynomial time algorithms to FPTAS using rounding and scaling techniques, builds upon a natural iterative process that extends and merges sets of node labels representing partial solutions, while keeping them small by discarding some solutions in an error controllable way.

1.2 Applications

We consider four problems that play a key role in several domains, including QoS routing in communication networks, traffic equilibria, transport optimization, and information dissemination.

Multiple Constrained (Optimal) Paths. The continuous demand for multimedia applications over the Internet has initiated a bulk of research on how to satisfy the QoS requirements of these applications (e.g., bandwidth, delay, jitter, packet loss, reliability, etc) [22]. One of the key issues in providing QoS guarantees is how to determine paths that satisfy QoS constraints, a problem known as QoS routing or constraint-based routing. The two most fundamental problems in QoS routing are

the *multiple constrained optimal path* (MCOP) and the *multiple constrained path* (MCP) problems (see e.g., [16, 20] and [22] for a survey). In MCOP, we are given a d-vector of costs $\mathbf{c}$ on the edges and a $(d-1)$-vector $\mathbf{b}$ of QoS-bounds. The objective is to find an s-t path p that minimizes $c_d(p) = \sum_{e \in p} c_d(e)$, and obeys the QoS-bounds, i.e., $c_i(p) = \sum_{e \in p} c_i(e) \leq b_i$, $\forall 1 \leq i \leq d-1$. MCOP is NP-hard, even when $d = 2$ in which case it is known as the restricted shortest path problem and admits a FPTAS [8, 18, 21]. In MCP, the objective is to find an s-t path p that simply obeys a d-vector $\mathbf{b}$ of QoS-bounds, i.e., $c_i(p) = \sum_{e \in p} c_i(e) \leq b_i$, $\forall 1 \leq i \leq d$. MCP is NP-complete. For both problems, the case of $d = 2$ objectives has been extensively studied and there are also very efficient FPTAS known [8, 21]. For $d > 2$, apart from the generic approach in [23], only heuristic methods and pseudopolynomial time algorithms are known [22]. In Section 4.1, we show how (quality guaranteed) approximate schemes to both MCOP and MCP can be constructed that have the same complexity with MOSP, thus improving upon previous approaches for any $d > 2$.

Non-Additive Shortest Paths. In this problem (NASP), we are given a digraph whose edges are associated with d-dimensional cost vectors and the task is to find a path that minimizes a certain d-attribute non-linear cost function. NASP is a fundamental problem in several domains [12, 13, 26], the most prominent of which is finding traffic equilibria [12, 26]. In such applications, a utility function for a path is defined that typically translates edge attributes (e.g., travel time, cost, distance, tolls, etc) to a common utility cost measure (e.g., money). Experience shows that users of traffic networks value certain attributes (e.g., time) non-linearly [19]: small amounts have relatively low value, while large amounts are very valuable. Also, the vast majority of toll road or transit systems have a non-additive (non-linear) toll/fare structure [12]. Consequently, the most interesting theoretical models for traffic equilibria [12, 26] involve minimizing a monotonic non-linear utility function. NASP is an NP-hard problem, and despite its importance no FPTAS is known. Previous approaches either deal with exact solutions, thus resulting in pseudopolynomial algorithms, or concern the efficient solution of the Lagrangian relaxation of NASP (best point in the convex hull of the Pareto set); see [27] and the discussion in that paper.

In Section 4.2, we show how our FPTAS for MOSP can be used to obtain a FPTAS for NASP. The complexity bound for NASP is identical to that for MOSP for a rather general form of the utility function that includes *any* polynomial of bounded degree with non-negative coefficients. This constitutes the *first* FPTAS for NASP.

Non-linear Objectives. Our solution machinery for NASP allows us to provide two general results for *any* multiobjective optimization problem. In particular, we show (Section 4.3) that: (i) given any multiobjective optimization problem along with its FPTAS, we can construct a FPTAS for its normalized version when the utility function is any polynomial of bounded degree with non-negative coefficients; (ii) a FPTAS for any multiobjective optimization problem with non-linear objectives, of the aforementioned form, can be constructed from a FPTAS for a much simpler version of the problem (with the same feasible solution set and objectives given by the *attributes* of the non-linear objective functions in the original problem). Since GAP routines for non-linear objectives are not known, this constitutes the first generic method for obtaining FPTAS for any multiobjective optimization problem with such non-linear objectives.

QoS-aware Multicommodity Flow. In the classical weighted multicommodity flow (MCF) problem, demands and commodity values (that multiply the flow in the objective function) are considered fixed. In several realistic network design scenarios, however, encountered in communication and transportation networks [1, 4, 5, 11, 29], this may not be the case, since demands and values are usually *elastic* to certain parameters – typically to the QoS provided by the network. We call this generalized version of the weighted MCF problem as *QoS-aware MCF*. Consider, for

instance, network operators in a public transportation network who wish to route various commodities (customers with common origin-destination pairs) to meet certain demands [24, 29]. It has been observed that when a customer is provided with a non-optimal path (route) due to unavailable capacity, s/he will most likely switch to another operator or even other means of transport and the probability in doing so increases as the QoS drops. To minimize the loss of customers, the value charged for the requested service is usually reduced to make the alternative (worse in QoS) path attractive. A similar situation is encountered with the aforementioned multimedia applications over the Internet or with information dissemination over various communication networks [5]. In such a setting, a "server" (owned by some service provider) sends information to "clients", who retrieve answers to queries they have posed regarding various types of information (or service). Common queries are typically grouped together. Answering a query incurs a cost and a data acquisition time that depends on the communication capacity. When a "client" is provided with a non-optimal service (e.g., long data acquisition time due to capacity restrictions), s/he will most likely switch to another provider. On the other hand, the provider may reduce the cost of such a service in order to minimize the loss. Consequently, network operators or service providers are confronted with the following design issues: which is the maximum profit obtained with the current capacity policy that incurs certain QoS-elastic demands and values? How much will this profit improve if the capacity is increased? Which is the necessary capacity to achieve a profit above a certain threshold? A fast algorithm for the QoS-aware MCF problem would allow network designers to address effectively such issues by identifying capacity bottlenecks and proceed accordingly.

We show (Section 4.4) that the QoS-aware MCF problem can be described as a fractional packing LP, and as such can be approximately solved by a Lagrangian-relaxation method like those in [10, 14, 25, 31] (see [2] for a comprehensive survey). For our purposes, we use the remarkably elegant and simple method given by Garg and Könemann [14] (henceforth the GK approach), combined with the phases technique by Fleischer [10], which assumes the existence of an oracle that identifies the most violated constraint of the dual LP. While in the classical weighted MCF problem the construction of the oracle reduces to the standard (single objective) shortest path problem, in the case of the QoS-aware MCF problem the oracle reduces to a multiobjective (actually non-additive) shortest path problem due to the QoS-elastic demands and values. Using our FPTAS, we can construct the required oracle and hence provide a FPTAS for the QoS-aware MCF problem.

2 Preliminaries

Recall that an instance of a multiobjective optimization problem is associated with a set of feasible solutions Q and a d-vector function $\mathbf{f} = [f_1, \ldots, f_d]^T$ associating each feasible solution $q \in Q$ with a d-vector $\mathbf{f}(q)$. The *Pareto set or curve* $\mathcal{P}$ of Q is defined as the set of all undominated elements of Q. Given a vector of approximation ratios $\boldsymbol{\rho} = [\rho_1, \ldots, \rho_d]^T$ ($\rho_i \geq 1$, $1 \leq i \leq d$), a solution $p \in Q$ *$\boldsymbol{\rho}$-covers* a solution $q \in Q$ iff it is as good in each objective i by at least a factor ρ_i, i.e., $f_i(p) \leq \rho_i \cdot f_i(q)$, $1 \leq i \leq d$. A set $\Pi \subseteq Q$ is a *$\boldsymbol{\rho}$-cover* of Q iff for all $q \in Q$, there exists $p \in \Pi$ such that p $\boldsymbol{\rho}$-covers q (note that a $\boldsymbol{\rho}$-cover may contain dominated solutions). A $\boldsymbol{\rho}$-cover is also called *$\boldsymbol{\rho}$-Pareto set*. If all entries of $\boldsymbol{\rho}$ are equal to ρ, we also use the terms ρ-cover and ρ-Pareto set.

A *fully polynomial time approximation scheme* (FPTAS) for computing the Pareto set of an instance of a multiobjective optimization problem is a family of algorithms that, for any fixed constant $\varepsilon > 0$, contains an algorithm that always outputs an $(1+\varepsilon)$-Pareto set and runs in time polynomial in the size of the input and $\frac{1}{\varepsilon}$.

If $\mathbf{a} = [a_1, a_2, \cdots, a_d]^T$ is a d-dimensional vector and λ a scalar, then we denote by $\mathbf{a}^\lambda = [a_1^\lambda, a_2^\lambda, \cdots, a_d^\lambda]^T$. A vector with all its elements equal to zero is denoted by $\mathbf{0}$.

3 Single-Source Multiobjective Shortest Paths

In the multiobjective shortest path problem, we are given a digraph $G = (V, E)$ and a d-dimensional function vector $\mathbf{c} : E \to [\mathbb{R}^+]^d$ associating each edge e with a cost vector $\mathbf{c}(e)$. We extend the cost function vector to handle paths by extending the domain to the powerset of E, thus considering the function $\mathbf{c} : 2^E \to [\mathbb{R}^+]^d$, where the cost vector of a path p is the sum of the cost vectors of its edges, i.e., $\mathbf{c}(p) = \sum_{e \in p} \mathbf{c}(e)$. Given two nodes v and w, let $P(v, w)$ denote the set of all v-w paths in G. In the *multiobjective shortest path* problem, we are asked to compute the Pareto set of $P(v, w)$ w.r.t. $\mathbf{c}$. In the *single-source multiobjective shortest path* (SSMOSP) problem, we are given a node s and the task is to compute the Pareto sets of $P(s, v)$ w.r.t. $\mathbf{c}$, $\forall v \in V$.

Given a vector $\boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{d-1}]^T$ of error parameters ($\varepsilon_i > 0$, $1 \leq i \leq d-1$) and a source node s, in this section we present an algorithm that computes, for each node v, a $\boldsymbol{\rho}$-cover of $P(s, v)$, where $\boldsymbol{\rho} = [1 + \varepsilon_1, 1 + \varepsilon_2, \cdots, 1 + \varepsilon_{d-1}, 1]^T$. Note that we can be *exact* in one dimension (here w.l.o.g. the d-th one), without any impact on the running time. In the following, let $c_i^{min} \equiv \min_{e \in E} c_i(e)$, $c_i^{max} \equiv \max_{e \in E} c_i(e)$, and $C_i = \frac{c_i^{max}}{c_i^{min}}$, for all $1 \leq i \leq d$. Let also $P^i(v, w)$ denote the set of all v-w paths in G with no more than i edges; clearly, $P^{n-1}(v, w) \equiv P(v, w)$.

3.1 The SSMOSP algorithm

Our algorithm can be viewed as a generalization of the classical (label correcting) Bellman-Ford algorithm. Previous algorithms that follow such an approach [3, 6, 7] have an exponential time complexity, since they keep all undominated solutions (exponentially large sets of labels). The key idea of our method is that we can implement the label sets as arrays of polynomial size by relaxing the requirements for strict Pareto optimality to that of $\boldsymbol{\rho}$-covering.

We represent a path $p = (e_1, e_2, \cdots, e_{k-1}, e_k)$ by a label that is a tuple $(\mathbf{c}(p), \text{pred}(p), \text{lastedge}(p))$, where $\mathbf{c}(p) = \sum_{e \in p} \mathbf{c}(e)$ is the d-dimensional cost vector of the path, $\text{pred}(p) = \vec{q}$ is a pointer to the label of the subpath $q = (e_1, e_2, \cdots, e_{k-1})$ of p, and $\text{lastedge}(p) = e_k$ points to the last edge of p. An empty label is represented by $(\mathbf{0}, null, null)$, while a single edge path has a *null* pred pointer. This representation allows us to retrieve the entire path, without implicitly storing its edges, by following the pred pointers. Let $\mathbf{r} = [r_1, \ldots, r_{d-1}, 1]$ be a vector of approximation ratios. The algorithm proceeds in rounds. In each round i and for each node v the algorithm computes a set of labels Π_v^i, which is an $\mathbf{r}^i$-cover of $P^i(s, v)$. We implement these sets of labels using $(d-1)$-dimensional arrays $\Pi_v^i[0..\lfloor \log_{r_1}(nC_1) \rfloor, 0..\lfloor \log_{r_2}(nC_2) \rfloor, \cdots, 0..\lfloor \log_{r_{d-1}}(nC_{d-1}) \rfloor]$, and index these arrays using $(d-1)$-vectors. This is done by defining a function $\mathbf{pos} : 2^E \to [\mathbb{N}_0]^{d-1}$. For a path p, $\mathbf{pos}(p) = [\lfloor \log_{r_1} \frac{c_1(p)}{c_1^{min}} \rfloor, \lfloor \log_{r_2} \frac{c_2(p)}{c_2^{min}} \rfloor, \cdots, \lfloor \log_{r_{d-1}} \frac{c_{d-1}(p)}{c_{d-1}^{min}} \rfloor]^T$ gives us the position in Π_v^i corresponding to p. The definition of **pos** along with the fact that for any path p we have $c_i(p) \leq (n-1)c_i^{max}$, $\forall 1 \leq i \leq d$, justifies the size of the arrays.

Initially, $\Pi_v^0 = \emptyset$, for all $v \in V - \{s\}$, and Π_s^0 contains only the trivial empty path. For each round $i \geq 1$ and for each node v the algorithm computes Π_v^i as follows (see also Fig. 1). Initially, we set Π_v^i equal to Π_v^{i-1}. We then examine the incoming edges of v, one by one, and perform an *Extend-&-Merge* operation for each edge examined. An *Extend-&-Merge* operation takes as input an edge $e = (u, v)$ and the sets Π_u^{i-1} and Π_v^i. It extends all the labels in Π_u^{i-1} by e, and merges the resulting set of s-v paths with Π_v^i, while maintaining at most one label (the one with the smallest c_d cost) for each position of the Π_v^i array, thus keeping the size of the sets polynomially bounded. In particular, the *Extend-&-Merge* operation on an edge $e = (u, v)$ is implemented as follows. We iterate through all labels $p \in \Pi_u^{i-1}$ and extend each p by e forming a new label (path) $q = (\mathbf{c}(p) + \mathbf{c}(e), \vec{p}, e)$. We then insert q in the position $\mathbf{pos}(q) = [\lfloor \log_{r_1} \frac{c_1(q)}{c_1^{min}} \rfloor, \lfloor \log_{r_2} \frac{c_2(q)}{c_2^{min}} \rfloor, \cdots, \lfloor \log_{r_{d-1}} \frac{c_{d-1}(q)}{c_{d-1}^{min}} \rfloor]^T$ of Π_v^i, unless this position is already filled in with a label q' for which $c_d(q') \leq c_d(q)$.

```
SSMOSP(G, s, c, r){
  forall v ∈ V {Π_v^0 = ∅;}
  Π_s^0[0] = {(0, null, null)};
  for i = 1 to n − 1{
    forall v ∈ V{
      Π_v^i = Π_v^{i−1};
      forall e = (u, v) ∈ E
        Π_v^i = Extend-&-Merge(Π_v^i, Π_u^{i−1}, e);
    }
  }
}
```

```
function Extend-&-Merge(R, Q, e) {
  forall p ∈ Q {
    q = (c(p) + c(e), p⃗, e);
    pos(q) = [⌊log_{r_1} c_1(q)/c_1^min⌋, ··· , ⌊log_{r_{d−1}} c_{d−1}(q)/c_{d−1}^min⌋]^T;
    if R[pos(q)] = null or c_d(R[pos(q)]) > c_d(q) {
      R[pos(q)] = q;
    }
  }
  return R;
}
```

Figure 1: The SSMOSP algorithm

The following lemma establishes the correctness of our approach.

Lemma 1 *For all $v \in V$ and for all $i \geq 0$, after the i-th round Π_v^i $\mathbf{r}^i$-covers $P^i(s,v)$.*

Proof. It suffices to prove that for all $p \in P^i(s,v)$, there exists $q \in \Pi_v^i$ such that $c_\ell(q) \leq r_\ell^i c_\ell(p)$, $\forall 1 \leq \ell \leq d$. We prove this by induction.

For the basis of the induction ($i = 1$) consider a single edge path $p \equiv (e) \in P^1(s,v)$. At each round all incoming edges of v are examined and an *Extend-&-Merge* operation is executed for each edge. After the first round and due to the **if** condition of the *Extend-&-Merge* operation, position $\mathbf{pos}(p)$ of Π_v^1 contains a path q for which: (i) $\mathbf{pos}(q) = \mathbf{pos}(p)$; and (ii) $c_d(q) \leq c_d(p)$. From (i) it is clear that for all $1 \leq \ell \leq d-1$, we have $\lfloor \log_{r_\ell} \frac{c_\ell(q)}{c_\ell^{min}} \rfloor = \lfloor \log_{r_\ell} \frac{c_\ell(p)}{c_\ell^{min}} \rfloor$, and therefore $\log_{r_\ell} \frac{c_\ell(q)}{c_\ell^{min}} - 1 \leq \log_{r_\ell} \frac{c_\ell(p)}{c_\ell^{min}}$. This, along with (ii) and the fact that $r_d = 1$, implies that $c_\ell(q) \leq r_\ell c_\ell(p)$, $\forall 1 \leq \ell \leq d$.

For the induction step consider a path $p \equiv (e_1, e_2, \ldots, e_k = (u,v)) \in P^i(s,v)$, for some $k \leq i$. The subpath $p' \equiv (e_1, e_2, \ldots, e_{k-1})$ of p has at most $i-1$ edges and applying the induction hypothesis we get that there exists a path $q' \in \Pi_u^{i-1}$ such that $c_\ell(q') \leq r_\ell^{i-1} c_\ell(p')$, $1 \leq \ell \leq d$. Let now $\bar{q}$ be the concatenation of q' with edge e_k. Then, we have:

$$c_\ell(\bar{q}) \leq r_\ell^{i-1} c_\ell(p), \quad 1 \leq \ell \leq d \tag{1}$$

It is clear by our algorithm that during the *Extend-&-Merge* operation for edge e_k in the i-th round $\bar{q}$ was examined. Moreover, at the end of the i-th round and due to the **if** condition of the *Extend-&-Merge* operation, position $\mathbf{pos}(\bar{q})$ of Π_v^i contains a path q for which: (iii) $\mathbf{pos}(q) = \mathbf{pos}(\bar{q})$; and (iv) $c_d(q) \leq c_d(\bar{q})$. From (iii) it is clear that $\lfloor \log_{r_\ell} c_\ell(q) \rfloor = \lfloor \log_{r_\ell} c_\ell(\bar{q}) \rfloor$, $\forall\ 1 \leq \ell \leq d-1$, and therefore $\log_{r_\ell} c_\ell(q) - 1 \leq \log_{r_\ell} c_\ell(\bar{q})$, $\forall\ 1 \leq \ell \leq d-1$, which implies that

$$c_\ell(q) \leq r_\ell c_\ell(\bar{q}), \quad 1 \leq \ell \leq d-1. \tag{2}$$

Since $r_d = 1$, combining now (iv) and (2) with (1), we get that $c_\ell(q) \leq r_\ell^i c_\ell(p)$, $\forall\ 1 \leq \ell \leq d$. ∎

We now turn to the time complexity.

Lemma 2 *Algorithm SSMOSP computes, for all $v \in V$, an $\mathbf{r}^{n-1}$-cover of $P(s,v)$ in total time $O(nm \prod_{j=1}^{d-1}(\lfloor \log_{r_j}(nC_j) \rfloor + 1))$.*

Proof. From Lemma 1, it is clear that, for any $v \in V$, Π_v^{n-1} is an $\mathbf{r}^{n-1}$-cover of $P^{n-1}(s,v) \equiv P(s,v)$, since any path has at most $n-1$ edges. The algorithm terminates after $n-1$ rounds. In each

round it examines all of the m edges and performs an *Extend-&-Merge* operation. The time of this operation is proportional to the size of the arrays used, which equals $\prod_{j=1}^{d-1}(\lfloor \log_{r_j}(nC_j)\rfloor + 1)$ and therefore the total time complexity is $O(nm\prod_{j=1}^{d-1}(\lfloor \log_{r_j}(nC_j)\rfloor + 1))$. ∎

Applying Lemma 2 with $\mathbf{r} = [(1+\varepsilon_1)^{\frac{1}{n-1}}, (1+\varepsilon_2)^{\frac{1}{n-1}}, \cdots, (1+\varepsilon_{d-1})^{\frac{1}{n-1}}, 1]$, and taking into account that $\ln(1+\delta) \approx \delta$ for small δ, yields the main result of this section.

Theorem 1 *Given a vector $\boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{d-1}]^T$ of error parameters and a source node s, there exists an algorithm that computes, for all $v \in V$, a $\boldsymbol{\rho}$-cover of $P(s,v)$ (set of all s-v paths), where $\boldsymbol{\rho} = [1+\varepsilon_1, 1+\varepsilon_2, \cdots, 1+\varepsilon_{d-1}, 1]^T$, in total time $O(n^d m \prod_{j=1}^{d-1}(\frac{1}{\varepsilon_j}\log(nC_j)))$.*

Let $C^{max} = \max_{1\le j\le d-1} C_j$. In the special case, where $\varepsilon_i = \varepsilon$, $\forall 1 \le i \le d-1$, we have the following result.

Corollary 1 *For any error parameter $\varepsilon > 0$, there exists a FPTAS for the single-source multiobjective shortest path problem with d objectives on a digraph G that computes $(1+\varepsilon)$-Pareto sets (one for each node of G) in total time $O(nm(\frac{n\log(nC^{max})}{\varepsilon})^{d-1})$.*

3.2 Extensions.

Further improvements can be obtained in the case of DAGs by exploiting the topological ordering of such graphs. In particular for each node v we maintain a set Π_v as in the general algorithm. Initially $\Pi_v = \emptyset$, $\forall v \in V - \{s\}$ and $\Pi_v[\mathbf{0}] = \{(\mathbf{0}, null, null)\}$. The algorithm visits all the nodes w.r.t. the topological order and for each visited node, it performs an *Extend-&-Merge* operation to all its outgoing edges. We can show that in this case the time reduces by a factor of n in all the aforementioned results. For instance, the time of Corollary 1 becomes $O(m(\frac{n\log(nC^{max})}{\varepsilon})^{d-1})$.

It is also quite easy to see that the algorithm actually computes an approximate Pareto curve w.r.t. the additional objective of minimizing the number of hops (number of edges in the path). Indeed, Lemma 1 implies that for any $v \in V$ the union of all Π_v^i, over all i rounds, constitutes an approximate Pareto curve also w.r.t. that additional objective.

4 Applications

We show how the result of Theorem 1 can be used to provide efficient approximate solutions to the MCOP, MCP, NASP, and QoS-aware MCF problems mentioned in the Introduction.

4.1 Multiple Constrained (Optimal) Paths

Let $\boldsymbol{\rho} = [1+\varepsilon_1, 1+\varepsilon_2, \cdots, 1+\varepsilon_{d-1}, 1]^T$ and let Π be a $\boldsymbol{\rho}$-cover Π of $P(s,t)$, constructed using the SSMOSP algorithm as implied by Theorem 1. For MCOP, choose $p' = \text{argmin}_{p\in\Pi}\{c_d(p); c_i(p) \le (1+\varepsilon_i)b_i,\ \forall 1 \le i \le d-1\}$. This provides a so-called *acceptable* solution in the sense of [16] by slightly relaxing the QoS-bounds; that is, the path p' is at least as good as the MCOP-optimum and is nearly feasible, violating each QoS-bound $1 \le i \le d-1$ by at most an $1+\varepsilon_i$ factor. For MCP, choose a path $p' \in \Pi$ that obeys the QoS-bounds, or answer that there is no path p in $P(s,t)$ for which $c_i(p) \le (1+\varepsilon_i)b_i$, $\forall 1 \le i \le d$. By Theorem 1, the required time for both cases is $O(n^d m \prod_{j=1}^{d-1}(\frac{1}{\varepsilon_j}\log(nC_j)))$, which can be reduced to $O(n^d m \prod_{j=1}^{d-1}(\frac{1}{\varepsilon_j}\log(\min\{nC_j, b_j/c_j^{min}\})))$ by observing that it is safe to discard any path p for which $c_j(p) > (1+\varepsilon_j)b_j$ for some $1 \le j \le d-1$ (thus reducing the size of the Π_v^i arrays).

4.2 Non-Additive Shortest Paths

In the *Non-Additive Shortest Path* (NASP) problem we are given a digraph $G = (V, E)$ and a d-dimensional function vector $\mathbf{c} : E \to [\mathbb{R}^+]^d$ associating each edge e with a vector of attributes $\mathbf{c}(e)$. A path p is also associated with a vector of attributes and $\mathbf{c}(p) = \sum_{e \in p} \mathbf{c}(e)$. We are also given a d-attribute non-decreasing and *non-linear* cost function $\mathcal{U} : [\mathbb{R}^+]^d \to \mathbb{R}$. The objective is to find a path p^*, from a specific source node s to a destination t, that minimizes the objective function, i.e., $p^* = \text{argmin}_{p \in P(s,t)} \mathcal{U}(\mathbf{c}(p))$. (It is easy to see that in the case where $\mathcal{U}$ is linear, NASP reduces to the classical single-objective shortest path problem.) For the general case of non-linear $\mathcal{U}$, it is not difficult to see that NASP is NP-hard (via a reduction from the restricted shortest path problem).

To obtain our FPTAS, we consider a quite general family of non-linear functions $\mathcal{U}(\mathbf{x})$. Specifically, let $\mathcal{U}(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \mathcal{G}_j(x_d) \prod_{k=1}^{d-1} {x_k}^{\beta_{jk}}$, where $\alpha_j, \beta_{jk} \geq 0$, and each $\mathcal{G}_j$ is any non-negative and non-decreasing function. Typically, N, α_j, and β_{jk} are considered constants, and we assume that the value of $\mathcal{U}(\mathbf{x})$, for any $\mathbf{x}$, can be computed in constant time. We can prove the following.

Theorem 2 *Let $\mathcal{U}(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \mathcal{G}_j(x_d) \prod_{k=1}^{d-1} {x_k}^{\beta_{jk}}$ be a cost function, where $\alpha_j, \beta_{jk} \geq 0$, and $\mathcal{G}_j$ be any non-negative and non-decreasing functions. Then, for any $\varepsilon > 0$, there exists an algorithm that computes a $(1+\varepsilon)$-approximation to the NASP optimum with respect to $\mathcal{U}(\mathbf{x})$ in time $O(n^d m(\frac{\log(nC^{max})\Delta}{\varepsilon})^{d-1})$, where $\Delta = \max_{1 \leq j \leq N} \sum_{k=1}^{d-1} \beta_{jk}$.*

Proof. Apply Theorem 1 and construct a $\boldsymbol{\rho}$-Pareto set Π of $P(s,t)$, with $\rho_i = (1+\varepsilon)^{\frac{1}{\Delta}}$, $\forall 1 \leq i \leq d-1$, and $\rho_d = 1$. Pick $p' = \text{argmin}_{p \in \Pi}(\mathcal{U}(\mathbf{c}(p)))$. Let p^* denote the optimal solution.

By the definition of Π we know that there exists some $q \in \Pi$ such that $c_k(q) \leq (1+\varepsilon)^{\frac{1}{\Delta}} c_k(p^*)$, $\forall 1 \leq k \leq d-1$, and $c_d(q) \leq c_d(p^*)$. Since $\alpha_j, \beta_{jk} \geq 0$, $\forall j, k$, we get:

$$\begin{aligned}
\mathcal{U}(\mathbf{c}(q)) &= \sum_{j=1}^{N} \alpha_j \mathcal{G}_j(c_d(q)) \prod_{k=1}^{d-1} (c_k(q))^{\beta_{jk}} \leq \sum_{j=1}^{N} \alpha_j \mathcal{G}_j(c_d(p^*)) \prod_{k=1}^{d-1} ((1+\varepsilon)^{\frac{1}{\Delta}} c_k(p^*))^{\beta_{jk}} \\
&\leq \sum_{j=1}^{N} ((1+\varepsilon)^{\frac{1}{\Delta}})^{\sum_{k=1}^{d-1} \beta_{jk}} \alpha_j \mathcal{G}_j(c_d(p^*)) \prod_{k=1}^{d-1} (c_k(p^*))^{\beta_{jk}} \leq ((1+\varepsilon)^{\frac{1}{\Delta}})^{\Delta} \mathcal{U}(\mathbf{c}(p^*))
\end{aligned}$$

Since $p' = \text{argmin}_{p \in \Pi}(\mathcal{U}(\mathbf{c}(p)))$, we get $\mathcal{U}(\mathbf{c}(p')) \leq \mathcal{U}(\mathbf{c}(q)) \leq (1+\varepsilon)\mathcal{U}(\mathbf{c}(p^*))$. ∎

It is clear from Theorem 2 that we can still have a FPTAS for NASP, when N and β_{jk} are bounded by some polynomial in the size of the input, and $\mathcal{U}(\mathbf{x})$, $\forall \mathbf{x}$, can be computed in time polynomial in the size of the input. A straightforward application of Theorem 2 gives the following.

Corollary 2 *If the cost function is of the form $\mathcal{U}([x_1, x_2]^T) = x_1 \mathcal{G}_1(x_2) + \mathcal{G}_2(x_2)$, with $\mathcal{G}_1, \mathcal{G}_2$ non-negative and non-decreasing, then, for any $\varepsilon > 0$, there is an algorithm that computes an $(1+\varepsilon)$-approximation to the optimum of NASP in time $O(n^2 m \frac{\log(nC_1)}{\varepsilon})$.*

4.3 Non-linear Objectives

The machinery of Theorem 2 can be used to obtain further results for multiobjective optimization problems. Let $\mathcal{M}$ be (an instance of) a multiobjective optimization problem with set of feasible solutions Q and vector of objective functions $\mathbf{c} = [c_1, \ldots, c_d]^T$, associating each feasible solution $q \in Q$ with a d-vector of attributes $\mathbf{c}(q)$.

Let $\mathcal{N}$ be the normalized version of $\mathcal{M}$ w.r.t. a non-linear utility function $\mathcal{U} : [\mathbb{R}^+]^d \to \mathbb{R}$; i.e., the objective of $\mathcal{N}$ is $\min_{q \in Q} \mathcal{U}(\mathbf{c}(q))$. Arguing similarly to Theorem 2 we can prove the following.

Theorem 3 *Let the objective function of $\mathcal{N}$ be of the form $\mathcal{U}(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \prod_{k=1}^{d} {x_k}^{\beta_{jk}}$. If there exists a FPTAS for $\mathcal{M}$ with time complexity poly$(1/\varepsilon, m)$, then there exists a FPTAS for $\mathcal{N}$ with complexity poly$(\Delta/\varepsilon, m)$, where m is the input size of $\mathcal{M}$ and $\Delta = \max_{1\leq j\leq N} \sum_{k=1}^{d} \beta_{jk}$.*

Now, let $\mathcal{M}'$ be a multiobjective optimization problem, defined on the same with $\mathcal{M}$ set of feasible solutions Q, but having a vector of objective functions $\mathbf{U} = [U_1, \ldots, U_h]^T$ associating each $q \in Q$ with an h-vector $\mathbf{U}(q)$. These objective functions are defined as $U_i(q) = \mathcal{U}_i(\mathbf{c}(q))$, $1 \leq i \leq h$, where $\mathcal{U}_i : [I\!R^+]^d \to I\!R$ are non-linear, non-decreasing functions. We can show the following.

Theorem 4 *Let the objective functions of $\mathcal{M}'$ be of the form $\mathcal{U}_i(\mathbf{x}) = \sum_{j=1}^{N_i} \alpha_{ij} \prod_{k=1}^{d} {x_k}^{\beta_{ijk}}$. If there exists a FPTAS for $\mathcal{M}$ with time complexity poly$(1/\varepsilon, m)$, then there exists a FPTAS for $\mathcal{M}'$ with complexity poly$(\Delta/\varepsilon, m)$, where m is the input size of $\mathcal{M}$ and $\Delta = \max_{1\leq i\leq h} \max_{1\leq j\leq N_i} \sum_{k=1}^{d} \beta_{ijk}$.*

Proof. We construct an $(1+\varepsilon)^{\frac{1}{\Delta}}$-Pareto curve Π for $\mathcal{M}$ and show that Π constitutes an $(1+\varepsilon)$-Pareto curve for $\mathcal{M}'$. To see this, it suffices to prove that for all $q \in Q$, there exists $p \in \Pi$ such that $U_i(p) \leq (1+\varepsilon)U_i(q)$, $\forall 1 \leq i \leq h$.

By the construction of Π he have that for all $q \in Q$ there exists $p \in \Pi$ such that $c_k(p) \leq (1+\varepsilon)^{\frac{1}{\Delta}} c_k(q)$, $\forall 1 \leq k \leq d$. Since $\beta_{ijk} \geq 0$, we have that

$$(c_k(p))^{\beta_{ijk}} \leq (1+\varepsilon)^{\frac{\beta_{ijk}}{\Delta}} (c_k(q))^{\beta_{ijk}}, \quad \forall 1 \leq i \leq h, 1 \leq j \leq N_i, 1 \leq k \leq d.$$

Multiplying over all k we get

$$\prod_{k=1}^{d} (c_k(p))^{\beta_{ijk}} \leq \prod_{k=1}^{d} (1+\varepsilon)^{\frac{\beta_{ijk}}{\Delta}} \prod_{k=1}^{d} (c_k(q))^{\beta_{ijk}}, \quad \forall 1 \leq i \leq h, 1 \leq j \leq N_i.$$

Taking the weighted sum over all j gives us

$$\sum_{j=1}^{N_i} \alpha_{ij} \prod_{k=1}^{d} (c_k(p))^{\beta_{ijk}} \leq \prod_{k=1}^{d} (1+\varepsilon)^{\frac{\beta_{ijk}}{\Delta}} \sum_{j=1}^{N_i} \alpha_{ij} \prod_{k=1}^{d} (c_k(q))^{\beta_{ijk}}, \quad \forall 1 \leq i \leq h.$$

By definition, $\sum_{k=1}^{d} \beta_{ijk} \leq \Delta$, $\forall 1 \leq i \leq h$, $1 \leq j \leq N_i$, and therefore

$$\sum_{j=1}^{N_i} \alpha_{ij} \prod_{k=1}^{d} (c_k(p))^{\beta_{ijk}} \leq (1+\varepsilon) \sum_{j=1}^{N_i} \alpha_{ij} \prod_{k=1}^{d} (c_k(q))^{\beta_{ijk}}, \quad \forall 1 \leq i \leq h,$$

and thus $U_i(p) \leq (1+\varepsilon)U_i(q)$, $\forall 1 \leq i \leq h$. ∎

4.4 QoS-aware Multicommodity Flow Problem

4.4.1 Problem Definition

We are given a digraph $G = (V, E)$, along with a capacity function $u : E \to I\!R_0^+$ on its edges. We are also given a set of k commodities. A commodity i, $1 \leq i \leq k$, is a tuple $(s_i, t_i, d_i, wt_i(\cdot), \mathrm{f_i}(\cdot), \mathrm{v_i}(\cdot))$, whose attributes are defined as follows. Attributes $s_i \in V$ and $t_i \in V$ are the source and the target nodes, respectively, while $d_i \in I\!R_0^+$ is the demand of the commodity. The weight function $wt_i : E \to I\!R_0^+$ quantifies the *quality of service (QoS)* for commodity i (smaller weight means better QoS). For any s_i-t_i path p, $wt_i(p) := \sum_{e\in p} wt_i(e)$ and let $\delta_i(s_i, t_i)$ be the length of the shortest path from s_i to t_i w.r.t. the weight function $wt_i(\cdot)$. The non-decreasing function $\mathrm{f_i} : [1, \infty) \to [0, 1]$ is the *elasticity function* of i that gives the portion $\mathrm{f_i}(x)$ of the commodity's demand d_i that we lose if the provided path is x times worse than the shortest path w.r.t $wt_i(\cdot)$. Commodity i is also

associated with a non-increasing *profit function* $\mathrm{v_i} : [1, \infty) \to \mathbb{R}_0^+$ that gives the profit $\mathrm{v_i}(x)$ from shipping one unit of flow of commodity i through a path that is x times worse than the shortest path w.r.t $wt_i(\cdot)$.

Let $P_i = \{p : p \text{ is an } s_i\text{-}t_i \text{ path}\}$ be the set of *candidate paths* along which flow from commodity i can be sent. Consider such a particular path $p \in P_i$ and let $X_i(p) \in \mathbb{R}_0^+$ denote the flow of commodity i routed along p. The definition of the elasticity function implies that for each unit of flow of commodity i routed along p, there are $\frac{1}{1-\mathrm{f_i}(x)}$ units *consumed* from the demand of the commodity. Thus, we define a *consumption function* $\mathrm{h_i} : [1, \infty) \to [1, \infty)$ with $\mathrm{h_i}(x) = \frac{1}{1-\mathrm{f_i}(x)}$. Since $\mathrm{f_i}$ is non-decreasing, $\mathrm{h_i}$ is also non-decreasing. Accordingly, we define the *consumption* $h_i(p) \geq 1$ of a path p as the amount of demand consumed for each unit of flow routed along p:

$$h_i(p) = \mathrm{h_i}\left(\frac{wt_i(p)}{\delta_i(s_i, t_i)}\right).$$

Similarly, we define the *value* $v_i(p)$ of a path p as the profit from routing one unit of flow of commodity i through p:

$$v_i(p) = \mathrm{v_i}\left(\frac{wt_i(p)}{\delta_i(s_i, t_i)}\right).$$

The objective is to maximize the total profit, which is the sum over all commodities of the flow routed from each commodity multiplied by the corresponding QoS-elastic value, subject to the capacity and demand constraints and w.r.t. the QoS-elasticity of the demands. To formulate the QoS-aware MCF problem as a linear program, for each commodity i and each path $p \in P_i$, we introduce a variable $X_i(p)$ denoting the flow of commodity i routed along p. Using the above definitions, the QoS-aware MCF problem can be described as follows.

$$\max \quad \sum_{i=1}^{k} \sum_{p \in P_i} v_i(p) X_i(p) \tag{3}$$

$$s.t. \quad \sum_{i=1}^{k} \sum_{e \in p, p \in P_i} X_i(p) \leq u(e), \forall e \in E \tag{4}$$

$$\sum_{p \in P_i} X_i(p) h_i(p) \leq d_i, \forall i = 1 \ldots k \tag{5}$$

4.4.2 Review of the GK Approach

A (pure) fractional packing LP is a linear program of the form $\max\{c^T x | Ax \leq b, x \geq 0\}$, where $A_{M \times N}$, $b_{M \times 1}$ and $c_{N \times 1}$ have positive entries. By scaling we also assume that $A(i, j) \leq b(i)$, $\forall i, j$. The dual of that problem is $\min\{b^T y | A^T y \geq c, y \geq 0\}$. In [14], Garg and Könemann present a remarkably elegant and simple FPTAS for solving fractional packing LPs. Their algorithm maintains a primal and a dual solution. At each step they identify the most violated constraint in the dual and increase the corresponding primal variable, and the dual variables. The most violated constraint is identified by using an exact oracle.

The algorithm works as follows. Let the *length* of a column j with respect to the dual variables y be $\mathrm{length}_y(j) = \sum_i \frac{A(i,j)}{c(j)} y(i)$. Let $a(y)$ denote the length of the minimum-length column, i.e., $a(y) = \min_j \mathrm{length}_y(j)$. Let also $D(y) = b^T y$ be the dual objective value with respect to y. Then, the dual problem is equivalent to finding an assignment y that minimizes $\frac{D(y)}{a(y)}$. The procedure is iterative. Let y_{k-1} be the dual variables and f_{k-1} be the value of the primal solution at the beginning of the k-th iteration. The initial values of the dual variables are $y_0(i) = \delta / b(i)$, where δ is a constant to be chosen later, and the primal variables are initially zero. In the k-th iteration, a call to an

oracle is made that returns the minimum length column q of A, i.e., $\text{length}_{y_{k-1}}(q) = \alpha(y_{k-1})$. Let now $p = \text{argmin}_i \frac{b(i)}{A(i,q)}$ be the "minimum capacity" row. In this iteration, we increase the primal variable $x(q)$ by $\frac{b(p)}{A(p,q)}$, thus the primal objective becomes $f_k = f_{k-1} + c(q)\frac{b(p)}{A(p,q)}$. The dual variables are updated as

$$y_k(i) = y_{k-1}(i)\left(1 + \varepsilon \frac{b(p)/A(p,q)}{b(i)/A(i,q)}\right),$$

where $\varepsilon > 0$ is a constant depending on the desired approximation ratio. For brevity we denote $a(y_k)$ and $D(y_k)$ by $a(k)$ and $D(k)$, respectively. The procedure stops at the first iteration t such that $D(t) \geq 1$. The final primal solution constructed may not be feasible since some of the packing constraints may be violated. However, scaling the final value of the primal variables by $\log_{1+\varepsilon} \frac{1+\varepsilon}{\delta}$ gives a feasible solution (see Lemma 7 in the Appendix).

The above algorithm can be straightforwardly extended to work with an approximate oracle[1]. Simply, in the k-th iteration we call an oracle that returns an $(1+w)$-approximation of the minimum length column of A. If q is the column returned by the oracle, then we have that $\text{length}_{y_{k-1}}(q) \leq (1+w)a(y_{k-1})$. By working similarly to [14] and choosing $\delta = (1+\varepsilon)((1+\varepsilon)M)^{-1/\varepsilon}$, we can show the following theorem (whose proof is in the Appendix for the sake of completeness).

Theorem 5 *There is an algorithm that computes an $(1-\varepsilon)^{-2}(1+w)$-approximation to the packing LP after at most $M\lceil \log_{1+\varepsilon} \frac{1+\varepsilon}{\delta} \rceil = M\lceil \frac{1}{\varepsilon} \log_{1+\varepsilon} M \rceil$ iterations, where M is the number of rows.*

4.4.3 The FPTAS for QoS-aware Multicommodity Flows

Recall the LP formulation of the QoS-aware MCF problem. To obtain its dual, we introduce for each edge e a dual variable $l(e)$ that corresponds to the capacity constraint (4) on e, and for each commodity i we introduce a dual variable ϕ_i that corresponds to the demand constraint (5) on i. The dual LP becomes

$$\max \quad D = \sum_{e \in E} l(e)u(e) + \sum_{i=1}^{k} \phi_i d_i \tag{6}$$

$$s.t. \quad l(p) + \phi_i h_i(p) \geq v_i(p), \forall i = 1 \ldots k, \forall p \in P_i \tag{7}$$

where $l(p) := \sum_{e \in p} l(e)$. It can be easily seen that the primal is a (pure) fractional packing LP. We solve this problem using the GK approach, which boils down in constructing a suitable approximate oracle to identify the most violated constraint (7) of the dual. To apply the algorithm, we have to scale the capacities and demands by $\min\{\min_{e \in E} u(e), \min_{1 \leq i \leq k} \frac{d_i}{h_i^{max}}\}$, where $h_i^{max} = h_i(\frac{(n-1)\max_{e \in E} wt_i(e)}{\delta_i(s_i,t_i)})$ is an upper bound on the maximum possible value of $h_i(\cdot)$, so that $u(e) \geq 1$, $\forall e \in E$, and $d_i \geq h_i(p)$, $\forall 1 \leq i \leq k$, $p \in P_i$.

Given an assignment (l, ϕ) for the dual variables, the length of a dual constraint is defined as $\text{length}_{(l,\phi)}(i,p) = \frac{l(p)+\phi_i h_i(p)}{v_i(p)}$ and by $a(l,\phi) = \min_{1 \leq i \leq k} \min_{p \in P_i} \text{length}_{(l,\phi)}(i,p)$ we denote the length of the most violated constraint. The algorithm maintains a dual variable $l(e)$ for each edge e, initially equal to $\frac{\delta}{u(e)}$, and a dual variable ϕ_i for each commodity i, initially equal to $\frac{\delta}{d_i}$, where $\delta = (1+\varepsilon)((1+\varepsilon)(m+k))^{-\frac{1}{\epsilon}}$.

The algorithm proceeds in iterations. Initially all flows are zero. In each iteration, it makes a call to an oracle that returns a commodity i' and a path $p \in P_{i'}$ that approximately minimizes $\text{length}_{(l,\phi)}(i,q)$ over all $1 \leq i \leq k$ and $q \in P_i$; i.e., we have $\text{length}_{(l,\phi)}(i',p) \leq (1+\varepsilon)a(l,\phi)$. It then

[1]Such an extension of the GK approach to work with approximate oracles was known before [15], and its combination with the phases technique of Fleischer [10] for solving packing problems has been first observed by Young [31] for solving the more general case of mixed packing LPs.

QoS-MCF($G, u, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{d}, \boldsymbol{wt}, \boldsymbol{v}, \varepsilon$) {
  forall $e \in E$ { $l(e) = \frac{\delta}{u(e)}$ }
  for $i = 1$ to k { $\phi_i = \frac{\delta}{d_i}$ }
  for $i = 1$ to k { **forall** $e \in E$ { $X_i(e) = 0$ }}
  $D = (m + k)\delta$;
  for $i = 1$ to k { $p_i = \text{NASP}(G, s_i, t_i, l, wt_i, \varepsilon)$ }
  $\overline{a} = \frac{1}{1+\varepsilon} \min_{1 \le i \le k} \left\{ \frac{l(p_i)+\phi_i h_i(p_i)}{v_i(p_i)} \right\}$;
  $i = 1$;
  while $D \le 1$ {
    $(p, i, \overline{a}) = \text{QoS-MCF-oracle}(G, \boldsymbol{s}, \boldsymbol{t}, l, \boldsymbol{wt}, \boldsymbol{v}, \boldsymbol{\phi}, \varepsilon, i, \overline{a})$;
    $\Delta = \min\{\frac{d_i}{h_i(p)}, \min_{e \in p} u(e)\}$;
    $X_i(p) = X_i(p) + \Delta$;
    forall $e \in p$ **do** $l(e) = l(e)(1 + \varepsilon \frac{\Delta}{u(e)})$;
    $\phi_i = \phi_i(1 + \varepsilon \frac{\Delta h_i(p)}{d_i})$;
    $D = D + \varepsilon \Delta \frac{l(p)+\phi_i h_i(p)}{v_i(p)}$;
  }
  for $i = 1$ to k { **forall** $e \in E$ { $X_i(e) = X_i(e)/\log_{1+\varepsilon} \frac{1+\varepsilon}{\delta}$ }}
}

QoS-MCF-oracle($G, \boldsymbol{s}, \boldsymbol{t}, l, \boldsymbol{wt}, \boldsymbol{v}, \boldsymbol{\phi}, \varepsilon, j, \overline{a}$) {
  while true {
    for $i = j$ to k {
      $p = \text{NASP}(G, s_i, t_i, l, wt_i, \varepsilon)$;
      if $\frac{l(p)+\phi_i h_i(p)}{v_i(p)} \le \overline{a}(1 + \varepsilon)^2$
        return $(p, i, \overline{a})$;
    }
    $\overline{a} = \overline{a}(1 + \varepsilon)$; /* update rule for next phase */
  }
}

Figure 2: The approximation algorithm for the QoS-MCF problem.

augments $\Delta = \min\{\frac{d'_i}{h'_i(p)}, \min_{e \in p} u(e)\}$ units of flow from commodity i' through p and updates the corresponding dual variables by setting $l(e) = l(e)(1 + \varepsilon \frac{\Delta}{u(e)})$, $\forall e \in p$, and $\phi_{i'} = \phi_{i'}(1 + \varepsilon \frac{\Delta h_{i'}(p)}{d_{i'}})$. The algorithm terminates at the first iteration for which $D = \sum_{e \in E} l(e)u(e) + \sum_{i=1}^{k} \phi_i d_i > 1$, and scales the final flow by $\log_{1+\varepsilon} \frac{1+\varepsilon}{\delta}$.

We now proceed to describe the oracle. Observe that our task is to approximately minimize, overall $1 \le i \le k$ and $q \in P_i$, the function

$$\frac{l(q) + \phi_i h_i(q)}{v_i(q)} = \frac{l(q) + \phi_i \cdot \mathrm{h}_i\left(\frac{wt_i(q)}{\delta_i(s_i,t_i)}\right)}{\mathrm{v}_i\left(\frac{wt_i(q)}{\delta_i(s_i,t_i)}\right)}.$$

Note that for a fixed i, the above function is of the form required by Corollary 2 with $\mathbf{c} = [l, wt_i]^T$, $\mathcal{G}_1(x) = \frac{1}{\mathrm{v}_i\left(\frac{x}{\delta_i(s_i,t_i)}\right)}$ and $\mathcal{G}_2(x) = \phi_i \cdot \frac{\mathrm{h}_i\left(\frac{x}{\delta_i(s_i,t_i)}\right)}{\mathrm{v}_i\left(\frac{x}{\delta_i(s_i,t_i)}\right)}$. Consequently, for a fixed i we can make use of a non-additive shortest path routine $\bar{p} = \text{NASP}(G, s_i, t_i, l, wt_i, \varepsilon)$ that returns an s_i-t_i path $\bar{p}$ that approximately (within $(1 + \varepsilon)$) minimizes the above function, overall $q \in P_i$.

To efficiently implement the oracle, we do not call the NASP routine for every value of i. Instead, the oracle proceeds in phases (like in [10]), maintaining a lower bound estimation $\overline{a}$ of $a(l, \phi)$, initially equal to $\overline{a} = \frac{1}{1+\varepsilon} \min_{1 \le i \le k} \{\frac{l(p_i)+\phi_i h_i(p_i)}{v_i(p_i)} | p_i = \text{NASP}(G, s_i, t_i, l, wt_i, \varepsilon)\}$. In each phase, the oracle examines the commodities one by one by performing NASP computations. For each commodity i the oracle returns any path $p = \text{NASP}(G, s_i, t_i, l, wt_i, \varepsilon)\}$ for which $\frac{l(p)+\phi_i h_i(p)}{v_i(p)} \le \overline{a}(1 + \varepsilon)^2$. It then continues with commodity $i + 1$. After all k commodities are considered in a phase, we know that $a(l, \phi) \ge (1 + \varepsilon)\overline{a}$ and proceed to the next phase by setting $\overline{a} = (1 + \varepsilon)\overline{a}$. The pseudocodes of our algorithm and the oracle are given in Fig. 2.

To discuss correctness and time bounds, we start with the following lemma that establishes an

upper bound on the ratio of the lengths of the minimum length column at the start and the end of the GK algorithm.

Lemma 3 *Let $a(0)$ and $a(t)$ be the lengths of the minimum length column at the start and the end of the algorithm, respectively. Then, $\frac{a(t)}{a(0)} \leq \frac{1+\varepsilon}{\delta}$.*

Proof. By the initial values of the dual variables, we have $a(0) = \min_j \sum_i \frac{A(i,j)}{c(j)} y_0(i) = \delta \cdot \min_j \sum_i \frac{A(i,j)}{c(j)b(i)}$. Since now the algorithm stops at the first iteration t such that $D(t) > 1$ and the dual variables increase by at most $1+\varepsilon$ in each iteration, it holds that $D(t) \leq 1+\varepsilon$. Consequently, $\sum_i b(i)y_t(i) \leq 1+\varepsilon$, which implies that $y_t(i) \leq (1+\varepsilon)\frac{1}{b(i)}$, $\forall i$. Hence, $a(t) = \min_j \sum_i \frac{A(i,j)}{c(j)} y_t(i) \leq (1+\varepsilon) \cdot \min_j \sum_i \frac{A(i,j)}{c(j)b(i)} = \frac{1+\varepsilon}{\delta} a(0)$. ∎

The following lemma establishes the approximation guarantee for the oracle.

Lemma 4 *A call to the oracle returns an $(1+\varepsilon)^2$-approximation of the most violated constraint in the dual.*

Proof. Let $\overline{a}_j$ be the value of $\overline{a}$ during the j-th phase of the algorithm. It suffices to show that for all phases $j \geq 1$, $\overline{a}_j \leq a(l,\phi)$.

Initially ($j = 1$), we set $\overline{a}_1 = \frac{1}{1+\varepsilon} \min_{1\leq i\leq k} \left\{ \frac{l(p_i)+\phi_i h_i(p_i)}{v_i(p_i)} | p_i = \text{NASP}(G, s_i, t_i, l, wt_i, \varepsilon) \right\}$. By the definition of the NASP routine, we get $\overline{a}_1 \leq \frac{1}{1+\varepsilon} \min_{1\leq i\leq k} \left\{ (1+\varepsilon) \min_{p\in P_i} \frac{l(p)+\phi_i h_i(p)}{v_i(p)} \right\} = a(l,\phi)$.

For any subsequent phase $j > 1$, consider phase $j-1$. The oracle finishes the examination of a commodity i and proceeds with $i+1$ only when a call to $\text{NASP}(G, s_i, t_i, l, wt_i, \varepsilon)$ in phase $j-1$ returns a path p_i for which $\frac{l(p_i)+\phi_i h_i(p_i)}{v_i(p_i)} > \overline{a}_{j-1}(1+\varepsilon)^2$. This inequality and the definition of the NASP routine imply that at the end phase $j-1$, we have for each commodity i

$$\overline{a}_{j-1}(1+\varepsilon)^2 < (1+\varepsilon) \min_{p\in P_i} \frac{l(p) + \phi_i h_i(p)}{v_i(p)}.$$

Hence, by the definition of $a(l,\phi)$, and since $l(e)$ can only increase during the algorithm, at the end of the phase we have $\overline{a}_{j-1}(1+\varepsilon)^2 < (1+\varepsilon)a(l,\phi)$. Since $\overline{a}_j = \overline{a}_{j-1}(1+\varepsilon)$, we get $\overline{a}_j < a(l,\phi)$. ∎

To establish a bound on the time complexity of the algorithm, we need to count the number of NASP computations. Clearly, at most one NASP computation is needed per augmentation of flow. The rest of NASP computations (not leading to an augmentation) are bounded by k times the number of phases. The following lemma establishes a bound on the total number of phases.

Lemma 5 *The number of phases of algorithm QoS-MCF is bounded by $\lceil \frac{1}{\varepsilon} \log_{1+\varepsilon}(m+k) \rceil + 2$.*

Proof. Let $a(0)$ and $a(t)$ be the lengths of the most violated constraint at the start and the end of the algorithm, respectively. Let now $\overline{a}_j$ be the value of $\overline{a}$ during the j-th phase of the algorithm, and T be the last phase of the algorithm.

Initially, we set $\overline{a}_1 = \frac{1}{1+\varepsilon} \min_{1\leq i\leq k} \left\{ \frac{l(p_i)+\phi_i h_i(p_i)}{v_i(p_i)} | p_i = \text{NASP}(G, s_i, t_i, l, wt_i, \varepsilon) \right\}$ and by the definition of the NASP routine we get that $a(0) \leq (1+\varepsilon)\overline{a}_1$. From the proof of Lemma 4, we have that $\overline{a}_T \leq a(t)$, and from Lemma 3 we get that $a(t) \leq \frac{1+\varepsilon}{\delta} a(0)$. Combining the last three inequalities we get $\overline{a}_T \leq \frac{(1+\varepsilon)^2}{\delta} \overline{a}_1$. By the update rule for $\overline{a}$ on each phase, we have that $\overline{a}_T = \overline{a}_1(1+\varepsilon)^{T-1}$, and therefore $\overline{a}_1(1+\varepsilon)^{T-1} \leq \frac{(1+\varepsilon)^2}{\delta} \overline{a}_1$, which implies that $T \leq \log_{1+\varepsilon} \frac{(1+\varepsilon)^3}{\delta}$. Hence, the number of phases is bounded by $\lceil \log_{1+\varepsilon} \frac{(1+\varepsilon)^3}{\delta} \rceil = \lceil \frac{1}{\varepsilon} \log_{1+\varepsilon}(m+k) \rceil + 2$, since $\delta = (1+\varepsilon)((1+\varepsilon)(m+k))^{-\frac{1}{\varepsilon}}$. ∎

We are now ready for the main result of this section.

Theorem 6 *There is an algorithm that computes an $(1-\varepsilon)^{-2}(1+\varepsilon)^2$-approximation to the QoS-aware MCF problem in time $O((\frac{1}{\varepsilon})^3(m+k)\log(m+k)mn^2(\frac{1}{\varepsilon}\log(m+k)+\log(nU))$, where n is the number of nodes, m is the number of edges, k is the number of commodities, and $U = \frac{\max_{e\in E} u(e)}{\min_{e\in E} u(e)}$.*

Proof. From Theorem 5 (with $M = m+k$) and Lemma 4 we have that the algorithm computes an $(1-\varepsilon)^{-2}(1+\varepsilon)^2$-approximation to the optimal and terminates after at most $(m+k)\lceil\log_{1+\varepsilon}(m+k)\rceil$ augmentations. Since for each phase at most k NASP computations do not lead to an augmentation, we get from Lemma 5 that the oracle performs at most $k\lceil\frac{1}{\varepsilon}\log_{1+\varepsilon}(m+k)\rceil+2k$ NASP computations not leading to an augmentation. Therefore, the total number of NASP computations during an execution of the algorithm is $O((m+k)\log_{1+\varepsilon}(m+k))$.

A NASP computation is carried out in time $O(\frac{1}{\varepsilon}n^2 m\log(nL))$, where $L = \frac{\max_{e\in E} l(e)}{\min_{e\in E} l(e)}$. From the initialization of $l(e)$, and since they can only increase during the algorithm, it is clear that $\min_{e\in E} l(e) \geq \frac{\delta}{\max_{e\in E} u(e)}$. Since now the algorithm stops at the first iteration such that $\sum_{e\in E} l(e)u(e)+\sum_{i=1}^{k}\phi_i d_i > 1$ and the dual variables increase by at most $1+\varepsilon$ in each iteration, it holds that $\sum_{e\in E} l(e)u(e) + \sum_{i=1}^{k}\phi_i d_i \leq 1+\varepsilon$. Consequently at the end of the algorithm we have $l(e) \leq \frac{(1+\varepsilon)}{u(e)}$, $\forall e \in E$, and thus $\max_{e\in E} l(e) \leq \frac{1+\varepsilon}{\min_{e\in E} u(e)}$. Hence, $L \leq \frac{1+\varepsilon}{\delta}U$. By our choice of $\delta = (1+\varepsilon)((1+\varepsilon)(m+k))^{-\frac{1}{\varepsilon}}$, we have that $L \leq ((1+\varepsilon)(m+k))^{\frac{1}{\varepsilon}}U$, and hence the time required for a NASP computation is $O(\frac{1}{\varepsilon}mn^2(\frac{1}{\varepsilon}\log(m+k)+\log(nU)))$. Thus, we get an algorithm that computes an $(1-\varepsilon)^{-2}(1+\varepsilon)^2$-approximation to the QoS-aware MCF problem in time $O((\frac{1}{\varepsilon})^3(m+k)\log(m+k)mn^2(\frac{1}{\varepsilon}\log(m+k)+\log(nU))$, which is polynomial to the input and $\frac{1}{\varepsilon}$. ∎

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A APPENDIX

A.1 Proof of Theorem 5

The analysis is straightforward from [14]. We only consider an approximate oracle. In order to prove Theorem 5 we need the next two lemmata. In the first lemma, we establish a bound on the ratio of the optimal dual value to the primal objective value at the end of the algorithm.

Lemma 6 *Let $\beta = \min_y \frac{D(y)}{a(y)}$ be the optimal dual value and let t be the last iteration of the algorithm. The ratio of the optimal dual value to the primal objective value at the end of the algorithm is bounded by $\frac{\varepsilon(1+w)}{\ln(1/M\delta)}$.*

Proof. For each iteration $k \geq 1$ it is

$$\begin{aligned} D(k) &= \sum_i b(i)y_k(i) \\ &= \sum_i b(i)y_{k-1}(i) + \varepsilon\frac{b(p)}{A(p,q)}\sum_i A(i,q)y_{k-1}(i) \\ &\leq D(k-1) + (1+w)\varepsilon(f_k - f_{k-1})a(k-1) \end{aligned}$$

which implies that

$$D(k) \leq D(0) + (1+w)\varepsilon\sum_{l=1}^{k}(f_l - f_{l-1})a(l-1).$$

Since $\beta = \min_y D(y)/a(y)$ it is $\beta \leq D(l-1)/a(l-1), \forall l = 1 \ldots k$, and thus

$$D(k) \leq M\delta + \frac{(1+w)\varepsilon}{\beta}\sum_{l=1}^{k}(f_l - f_{l-1})D(l-1).$$

Observe now that for fixed k, this right hand side is maximized by setting $D(l-1)$ to its maximum possible value for all $1 \leq l-1 < k$, and let us denote this maximum value by $D'(k)$, i.e.,

$$D'(k) = M\delta + \frac{(1+w)\varepsilon}{\beta}\sum_{l=1}^{k}(f_l - f_{l-1})D'(l-1).$$

Consequently,

$$\begin{aligned} D(k) &\leq D'(k) \\ &= D'(k-1) + \frac{(1+w)\varepsilon}{\beta}(f_k - f_{k-1})D'(k-1) \\ &= D'(k-1)\left(1 + \frac{(1+w)\varepsilon}{\beta}(f_k - f_{k-1})\right) \\ &\leq D'(k-1)e^{\frac{(1+w)\varepsilon}{\beta}(f_k - f_{k-1})} \\ &\leq D'(0)e^{\frac{(1+w)\varepsilon}{\beta}\sum_{l=1}^{k}(f_l - f_{l-1})} \\ &\leq D'(0)e^{\frac{(1+w)\varepsilon}{\beta}(f_k - f_0)} \end{aligned}$$

Since now $D'(0) = M\delta$ and $f_0 = 0$ it follows that

$$D(k) \leq M\delta e^{(1+w)\varepsilon f_k/\beta}.$$

From our stopping condition it is $1 \leq D(t) \leq M\delta e^{(1+w)\varepsilon f_t/\beta}$ and hence

$$\frac{\beta}{f_t} \leq \frac{\varepsilon(1+w)}{\ln(1/M\delta)}.$$

∎

The final primal solution constructed may not be feasible since some of the packing constraints may be violated. The second lemma shows that the final primal assignment can be appropriately scaled as to obtain a feasible solution.

Lemma 7 *Scaling the final primal assignment by* $\log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right)$*, we obtain a feasible solution to the fractional packing LP.*

Proof. When we pick a column q and increase the left-hand-side of the i-th constraint by $\frac{A(i,q)b(p)}{A(p,q)b(i)}$. Simultaneously we increase the dual variable $y(i)$ by a multiplicative factor of $1 + \varepsilon\frac{A(i,q)b(p)}{A(p,q)b(i)}$. By the definition of p it follows that $\frac{A(i,q)b(p)}{A(p,q)b(i)} \leq 1$ and thus increasing the left-hand-side of the i-th constraint by one causes an increase in $y(i)$ by a multiplicative factor of $1 + \varepsilon$. Since t is the first iteration for which $D(t) > 1$, it is $y_{t-1}(i) < 1/b(i)$ and thus $y_t(i) < (1+\varepsilon)/b(i)$. Since now $y_0(i) < \delta/b(i)$ it follows that the left-hand-side of the i-th constraint is no more than $\log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right)$ for any i. Thus scaling the primal solution by $\log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right)$ gives a feasible solution. ∎

We now proceed with the proof of Theorem 5.

Proof of Theorem 5: In the k-th iteration we increase the dual variable of the "minimum capacity" row by a factor of $1 + \varepsilon$. Since we stop the algorithm at the first iteration t such that $D(t) > 1$ it follows that $D(t) < 1 + \varepsilon$ and thus $y_t(i) < \frac{1+\varepsilon}{b(i)}$ for any row. Since now $y_0(i) = \frac{\delta}{b(i)}$ and $y_t(i) < \frac{1+\varepsilon}{b(i)}$ and there are M rows the total number of iterations is at most $M\lceil\log_{1+\varepsilon}\frac{1+\varepsilon}{\delta}\rceil = M\lceil\frac{1}{\varepsilon}\log_{1+\varepsilon}M\rceil$, by choosing $\delta = (1+\varepsilon)((1+\varepsilon)M)^{-1/\varepsilon}$.

The ratio of the optimal dual value to objective value of the scaled final primal assignment is $\gamma = \frac{\beta}{f_t}\log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right)$ by substituting the bound on $\frac{\beta}{f_t}$ from Lemma 6 we get

$$\gamma \leq \frac{\varepsilon(1+w)}{\ln(1/M\delta)}\log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right) = \frac{\varepsilon(1+w)}{\ln(1+\varepsilon)}\frac{\ln\frac{1+\varepsilon}{\delta}}{\ln(1/M\delta)}$$

For $\delta = (1+\varepsilon)((1+\varepsilon)M)^{-1/\varepsilon}$, the ratio $\frac{\ln \frac{1+\varepsilon}{\delta}}{\ln(1/M\delta)}$ equals $(1-\varepsilon)^{-1}$, hence we have

$$\gamma \leq \frac{\varepsilon(1+w)}{(1-\varepsilon)\ln(1+\varepsilon)} \leq \frac{\varepsilon(1+w)}{(1-\varepsilon)(\varepsilon-\varepsilon^2/2)} \leq (1-\varepsilon)^{-2}(1+w)$$

■