Evaluating approval-based multiwinner voting in terms of robustness to noise*

Ioannis Caragiannis† Christos Kaklamanis ‡ Nikos Karanikolas§
George A. Krimpas¶

Abstract

Approval-based multiwinner voting rules have recently received much attention in the Computational Social Choice literature. Such rules aggregate approval ballots and determine a winning committee of alternatives. To assess effectiveness, we propose to employ new noise models that are specifically tailored for approval votes and committees. These models take as input a ground truth committee and return random approval votes to be thought of as noisy estimates of the ground truth. A minimum robustness requirement for an approval-based multiwinner voting rule is to return the ground truth when applied to profiles with sufficiently many noisy votes. Our results indicate that approval-based multiwinner voting is always robust to reasonable noise. We further refine this finding by presenting a hierarchy of rules in terms of how robust to noise they are.

1 Introduction

Voting has received much attention by the AI community recently, mostly due to its suitability for simple and effective decision making. One popular line of research, that originates from Arrow [1], has aimed to characterize voting rules in terms of the *social choice axioms* they satisfy. Another approach views voting rules as *estimators*. It assumes that there is an objectively correct choice, a *ground truth*, and votes are noisy estimates of it. Then, the main criterion for evaluating a voting rule is whether it can determine the ground truth as outcome when applied to noisy votes.

A typical scenario in studies that follow the second approach employs a hypothetical *noise model* that uses the ground truth as input and produces random votes. Then, a voting rule is applied on profiles of such random votes and is considered effective if it acts as a *maximum likelihood estimator* [10, 24] or if it has *low sample complexity* [8]. As such evaluations are heavily dependent on the specifics of the noise model, relaxed effectiveness requirements, such as the *accuracy in the limit*, sought in broad classes of noise models [8] can be more informative.

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†Computer Technology Institute "Diophantus" & Department of Computer Engineering and Informatics, University of Patras, 26504 Rion, Greece. Email: caragian@ceid.upatras.gr

‡Computer Technology Institute "Diophantus" & Department of Computer Engineering and Informatics, University of Patras, 26504 Rion, Greece. Email: kakl@ceid.upatras.gr

§Department of Computer Engineering and Informatics, University of Patras, 26504 Rion, Greece. Email: nkaranik@ceid.upatras.gr

¶Department of Computer Engineering and Informatics, University of Patras, 26504 Rion, Greece. Email: krimpas@ceid.upatras.gr

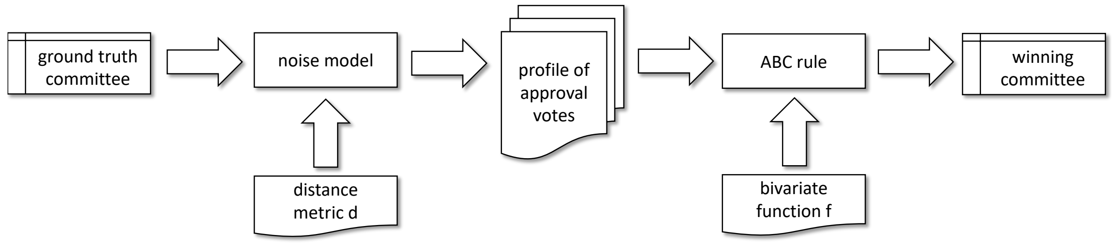

Figure 1: Our evaluation framework.

We restrict our attention to *approval voting*, where ballots are simply sets of alternatives that are approved by the voters [17]. Furthermore, we consider *multiwinner voting rules* [13], which determine committees of alternatives as outcomes [14, 2]. And, in particular, we focus on approval-based counting choice rules (or, simply, *ABC rules*), which were defined recently by Lackner and Skowron [15]. A famous rule in this category is known as multiwinner approval voting (AV). Each alternative gets a point every time it appears in an approval vote and the outcome consists of a fixed number of alternatives with the highest scores.

We consider noise models that are particularly tailored for approval votes and committees. These models use a committee as ground truth and produce random sets of alternatives as votes. We construct broad classes of noise models that share a particular structure, parameterized by *distance metrics* defined over sets of alternatives. In this way, we adapt to approval-based multiwinner voting the approach of Caragiannis et al. [8] for voting rules over rankings.

Figure 1 illustrates our evaluation framework. The noise model is depicted at the left. It takes as input the ground truth committee and its probability distribution over approval votes is consistent to a distance metric d. Repeated executions of the noise model produce a profile of random approval votes. The ABC rule (defined using a bivariate function f; see Section 2) is then applied on this profile and returns a winning committee. Our requirement for the ABC rule is to be accurate in the limit, not only for a single noise model, but for all models that belong to a sufficiently broad class. The breadth of this class quantifies the robustness of the ABC rule to noise.

The details of our framework are presented in Section 2. Our results indicate that it indeed allows for a classification of ABC rules in terms of their robustness to noise. In particular, we identify (in Section 3) the modal committee rule (MC) as the ultimately robust ABC rule: MC is robust against all kinds of reasonable noise. AV follows in terms of robustness and seems to outperform other known ABC rules (see Section 4). In contrast, the well-known approval Chamberlin-Courant (CC) rule is the least robust. On the other hand, all ABC rules are robust if we restrict noise sufficiently (see Section 5). We conclude with a discussion on open problems in Section 6.

1.1 Further related work

Approval-based multiwinner voting rules have been studied in terms of their computational complexity [3, 22], axiomatic properties [21, 15, 2], as well as their applications [5]. In particular, axiomatic work has focused on two different principles that govern multiwinner rules: diversity and individual excellence. Lackner and Skowron [16] attempt a quantification of how close an approval-based multiwinner voting rule is to these two principles. We remark that the primary focus of the current paper is on individual excellence.

The robustness of approval voting has been previously evaluated against noise models, using either the MLE [20] or the sample complexity [6] approach. These papers assume a ranking of the alternatives as ground truth, generate approval votes that consist of the top alternatives in rankings produced according to the noise model of Mallows [18], and assess how well approval voting recovers the ground truth ranking. We believe that our framework is fairer to approval votes, as recovering an underlying ranking when voters have very limited power to rank is very demanding. The robustness of multiwinner voting against noise has been studied by Procaccia et al. [19].

Additional references related to specific ABC rules are given in the next section. We remark that the modal committee (MC) rule is similar in spirit to the modal ranking rule considered by Caragiannis et al. [7].

2 Preliminaries

Throughout the paper, we denote by A the set of alternatives. We use $m = |A|$ and denote the committee size by k. The term committee refers to a set of exactly k alternatives.

Approval-based multiwinner voting. An approval vote is simply a subset of the alternatives (of any size). An approval-based multiwinner voting rule takes as input a profile of approval votes and returns one or more winning committees.

We particularly consider voting rules that belong to the class of approval-based counting choice rules (or, simply, ABC rules), introduced by Lackner and Skowron [15]. Such a rule is defined by a bivariate function f, with $f(x, y)$ indicating a non-negative score a committee gets from an approval vote containing y alternatives, x of which are common with the committee. f is non-decreasing in its first argument. Formally, f is defined on the set $\mathcal{X}_{m,k}$, which consists of all pairs (x, y) of possible values of $|U \cap S|$ and $|S|$, given that U is k-sized and S can be any subset of the m alternatives of A. I.e., $\mathcal{X}_{m,k}$ is the set

$$\{(x, y) : y = 0, 1, ..., m, x = \max\{k + y - m, 0\}, ..., y\}.$$

The score of a committee is simply the total score it gets from all approval votes in a profile. Winning committees are those that have maximum score. We extensively use "the ABC rule f" to refer to the ABC rule that uses the bivariate function f. We denote the score that an ABC rule f assigns to the committee U given a profile $\Pi = (S_i)_{i\in[n]}$ of n votes by $\mathrm{sc}_f(U, \Pi) = \sum_{i=1}^{n} f(|U \cap S_i|, |S|)$. With some abuse of notation, we use $\mathrm{sc}_f(U, S_i)$ to refer to the score U gets from vote S_i. Hence, $\mathrm{sc}_f(U, \Pi) = \sum_{i=1}^{n} \mathrm{sc}_f(U, S_i)$.

Well-known ABC rules include:

- Multiwinner approval voting (AV), which uses the function $f_{\mathrm{AV}}(x, y) = x$.
- Approval Chamberlin-Courant (CC), which uses the function $f_{\mathrm{CC}}(x, y) = \min\{1, x\}$. The rule falls within a more general context considered by Chamberlin and Courant [9].
- Proportional approval voting (PAV), which uses the function $f_{\mathrm{PAV}}(x, y) = \sum_{i=1}^{x} 1/i$.

These rules belong to the class of rules that originate from the work of Thiele [23]. A Thiele rule uses a vector $\langle w_1, w_2, ..., w_k\rangle$ of non-negative weights to define $f(x, y) = \sum_{i=1}^{x} w_j$. Other known Thiele rules include the p-Geometric rule [22] and Sainte Laguë approval voting [16].

A well-known non-Thiele rule is the satisfaction approval voting (SAV) rule that uses $f_{\mathrm{SAV}}(x, y) = x/y$ for $y > 0$ and $f(x, y) = 0$ otherwise [4]. Let us also introduce the *modal committee* (MC) rule which returns the committee that has maximum number of appearances as approval votes in the profile. MC is also non-Thiele; it uses $f(k, k) = 1$ and $f(x, y) = 0$ otherwise.

Noise models. We employ noise models to generate approval votes, assuming that the ground truth is a committee. Denoting the ground truth by $U \subseteq A$, a noise model $\mathcal{M}$ produces random approval votes according to a particular distribution that defines the probability $\Pr_{\mathcal{M}}[S|U]$ to generate the set $S \subseteq A$ when the ground truth is U.

Let us give the following noise model $\mathcal{M}_p$ as an example. $\mathcal{M}_p$ uses a parameter $p \in (1/2, 1]$. Given a ground truth committee U, $\mathcal{M}_p$ generates a random set $S \subseteq A$ by selecting each alternative of U with probability p and each alternative in $A \setminus U$ with probability $1 - p$. Intuitively, the probability that a set will be generated depends on its "distance" from the ground truth: the higher this distance, the smaller this probability. To make this formal, we will need the set difference distance metric $d_\Delta : 2^A \to \mathbb{R}_{\geq 0}$ defined as $d_\Delta(X, Y) = |X \setminus Y| + |Y \setminus X|$.

Claim 1. *For $S \subseteq A$, $\Pr_{\mathcal{M}_p}[S|U] = p^m \left(\frac{1-p}{p}\right)^{d_\Delta(U,S)}$.*

So, the probability $\Pr_{\mathcal{M}_p}[S|U]$ is decreasing in $d_\Delta(U, S)$. We will consider general noise models $\mathcal{M}$ with $\Pr_{\mathcal{M}}[S|U]$ depending on $d(U, S)$, where d is a distance metric defined over subsets of A.

Definition 1. *Let d be a distance metric over sets of alternatives. A noise model $\mathcal{M}$ is called d-monotonic if for any two sets $S_1, S_2 \subseteq A$, it holds $\Pr_{\mathcal{M}}[S_1|U] > \Pr_{\mathcal{M}}[S_2|U]$ if and only if $d(U, S_1) < d(U, S_2)$.*

Definition 1 implies that $\Pr_{\mathcal{M}}[S_1|U] = \Pr_{\mathcal{M}}[S_2|U]$ when $d(U, S_1) = d(U, S_2)$.

Besides the set difference metric used by $\mathcal{M}_p$, other well-known distance metrics[1] (see Deza and Deza [11]) are:

- the normalized set difference or Jaccard metric d_J, defined as $d_J(X, Y) = \frac{d_\Delta(X,Y)}{|X \cup Y|}$,
- the maximum difference or Zelinka metric d_Z, defined as $d_Z(X, Y) = \max\{|X \setminus Y|, |Y \setminus X|\}$, and
- the normalized maximum difference or Bunke-Shearer metric d_{BS}, defined as $d_{BS}(X, Y) = \frac{d_Z(X,Y)}{\max\{|X|,|Y|\}}$.

Evaluating ABC rules against noise models. We aim to evaluate the effectiveness of ABC rules when applied to random profiles generated by large classes of noise models. To this end, we use *accuracy in the limit* as a measure.

Definition 2 (accuracy in the limit)**.** *An ABC rule f is called* accurate in the limit *for a noise model $\mathcal{M}$ if there exists n_0 such that, for every profile of $n \geq n_0$ approval votes produced by $\mathcal{M}$ with ground truth U, f returns U as the unique winning committee with certainty.*

Then, ABC rules are evaluated in terms of robustness using the next definition.

Definition 3 (robustness)**.** *Let d be a distance metric over sets of alternatives. An ABC rule f is monotone robust against d (or d-monotone robust) if it is accurate in the limit for all d-monotonic noise models.*

[1]Notice that $d(X, Y)$ for the four specific distance metrics defined here depends only on $|X \setminus Y|$, $|Y \setminus X|$, $|X|$, and $|Y|$. In a sense, these distance metrics are *alternative-independent*. Our results apply to the most general definition of distance, where $d(X, Y)$ can also depend on the contents of $X \setminus Y$, $Y \setminus X$, X, and Y.

3 MC is a uniquely robust ABC rule

We begin our technical exposition by identifying the unique ABC rule that is monotone robust against *all* distance metrics. Our proofs, in the current and subsequent sections, make extensive use of the following lemma. The notation $S \sim \mathcal{M}(U)$ indicates that the random set S is drawn from the noise model $\mathcal{M}$ with ground truth U.

Lemma 2. *An ABC rule f is accurate in the limit for a noise model $\mathcal{M}$ if and only if $\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_f(U,S)-\mathrm{sc}_f(V,S)] > 0$ for every two different sets of alternatives $U,V \subseteq A$ with $|U|=|V|=k$.*

Proof. Let U be a k-sized set of alternatives. For integer $n \geq 1$, let Π_n be the profile that consists of approval votes S_1, S_2, ..., S_n that have been produced independently from the noise model $\mathcal{M}$ with ground truth U. We will show that, as n tends to infinity, U is a winning committee under f in Π_n with probability 1 if and only if $\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_f(U,S)-\mathrm{sc}_f(V,S)] > 0$.

Let V be any k-sized set of alternatives different from U. For $i=1,2,...$, let $X_i = \mathrm{sc}_f(U,S_i) - \mathrm{sc}_f(V,S_i)$. Now, for $n=1,2,..$, define

$$\begin{aligned} Y_n &= \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}\sum_{i=1}^{n}\left(\mathrm{sc}_f(U,S_i)-\mathrm{sc}_f(V,S_i)\right) \\ &= \frac{1}{n}\left(\mathrm{sc}_f(U,\Pi_n)-\mathrm{sc}_f(V,\Pi_n)\right). \end{aligned}$$

I.e., a positive Y_n implies that the committee U is superior to V for profile Π_n according to the ABC rule f. By the law of large numbers, we have

$$\lim_{n\to+\infty} Y_n = \mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_f(U,S)-\mathrm{sc}_f(V,S)]. \tag{1}$$

If the RHS of (1) is positive, then there is n_0 such that $Y_n > 0$ for every $n \geq n_0$ and, consequently, $\mathrm{sc}_f(U,\Pi_n) > \mathrm{sc}_f(V,\Pi_n)$. We conclude that, as n tends to infinity, U is (with certainty) the unique winning committee for Π_n according to f.

If the RHS of (1) is non-positive, linearity of expectation yields $\mathbb{E}[Y_n] \leq 0$ for every $n \geq 1$. This implies that

$$\Pr[\mathrm{sc}_f(U,\Pi_n) \leq \mathrm{sc}_f(V,\Pi_n)] = \Pr[Y_n \leq 0] > 0$$

and the probability that U is the unique winning committee for Π_n according to f is strictly smaller than 1. □

We are ready to present our first application of Lemma 2.

Theorem 3. *MC is the only ABC rule that is monotone robust against any distance metric.*

Proof. Let $\mathcal{M}$ be a noise model that is d-monotonic for some distance metric d. Let $U,V \subseteq A$ be any two different k-sized sets of alternatives. By the definition of MC, we have

$$\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_{\mathrm{MC}}(U,S)-\mathrm{sc}_{\mathrm{MC}}(V,S)] = \Pr_{\mathcal{M}}[U|U]-\Pr_{\mathcal{M}}[V|U] > 0.$$

By Lemma 2, we obtain that MC is d-monotone robust.

We will now show that MC is the only ABC rule f that has this property. Let f be an ABC rule that is different than MC. This means that there exist integers x^* and y^* with $(x^*-1,y^*),(x^*,y^*) \in \mathcal{X}_{m,k}$, $(x^*,y^*) \neq (k,k)$, and $f(x^*,y^*) > f(x^*-1,y^*)$. We will construct a distance metric d and a d-monotonic noise model for which f is not accurate in the limit.

Rename the alternatives of A as $a_1, a_2, ..., a_m$ and let $U = \{a_1, a_2, ..., a_k\}$, $V = \{a_2, ..., a_{k+1}\}$, and $W = \{a_{k-x^*+2}, ..., a_{y^*+k-x+1}\}$. Notice that, by the definition of $\mathcal{X}_{m,k}$, $(x^* - 1, y^*) \in \mathcal{X}_{m,k}$ implies that $1 + \max\{y^* + k - m, 0\} \leq x^*$ and, equivalently, $y^* + k - x^* + 1 \leq m$; hence, the set W is well-defined. Clearly, $x^* \geq 1$; so sets V and W share at least one alternative.

We define a distance metric between subsets of A that has $d(X, Y) = 0$ if $X = Y$, $d(X, Y) \in \{1, 2\}$, otherwise, and in particular $d(U, V) = d(U, W) = 1$ and $d(U, S) = 2$ for every S different than U, V, or W.

We are ready to define the d-monotonic noise model $\mathcal{M}$. For simplicity, we use $p_0 = \Pr_{\mathcal{M}}[U|U]$, $p_1 = \Pr_{\mathcal{M}}[V|U] = \Pr_{\mathcal{M}}[W|U]$, and $p_2 = \Pr_{\mathcal{M}}[S|U]$ for every other set $S \subseteq A$ different than U, V, or W. For $\delta > 0$ (to be specified shortly), we set $p_0 = 1/3$, $p_1 = 1/3 - \delta$, and $p_2 = \frac{2\delta}{2^m - 3}$.

We now compute the quantity $\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_f(U, S) - \mathrm{sc}_f(V, S)]$; observe that $\mathrm{sc}_f(U, U) = \mathrm{sc}_f(V, V) = f(k, k)$, $\mathrm{sc}_f(U, V) = \mathrm{sc}_f(V, U) = f(k - 1, k)$, $\mathrm{sc}_f(U, W) = f(x^* - 1, y^*)$, and $f(V, W) = f(x^*, y^*)$. We obtain

$$\begin{aligned}
&\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_f(U, S) - \mathrm{sc}_f(V, S)] \\
&= f(k, k)p_0 + f(k - 1, k)p_1 + f(x^* - 1, y^*)p_1 \\
&\quad + \sum_{S\neq U,V,W} f(|U \cap S|, |S|)p_2 - f(k - 1, k)p_0 \\
&\qquad - f(k, k)p_1 - f(x^*, y^*)p1 - \sum_{S\neq U,V,W} f(|V \cap S|, |S|)p_2 \\
&\leq (p_0 - p_1)(f(k, k) - f(k - 1, k)) - p_1(f(x^*, y^*) - f(x^* - 1, y^*)) \\
&\quad - p_2 \sum_{S\neq U,V,W} f(|V \cap S|, |S|) \\
&= \delta(f(k, k) - f(k - 1, k)) - (1/3 - \delta)(f(x^*, y^*) - f(x^* - 1, y^*)) \\
&\quad + \frac{2\delta}{2^m - 3} \sum_{S\neq U,V,W} f(|V \cap S|, |S|). \qquad (2)
\end{aligned}$$

Observe that the RHS of (2) is increasing in δ and approaches $-\frac{1}{3}(f(x^*, y^*) - f(x^* - 1, y^*)) < 0$ as δ approaches 0. Hence, for a sufficiently small positive δ, we have

$$\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_f(U, S) - \mathrm{sc}_f(V, S)] < 0.$$

By Lemma 2, f is not accurate in the limit for $\mathcal{M}$. □

4 A characterization for AV

In this section, we identify the class of distance metrics against which AV is monotone robust. Before defining this class, let us fix some notation; this will be useful in several proofs.

For a distance metric d and a set of alternatives U, let $\mathrm{span}(d, U)$ be the number of different non-zero values the quantity $d(U, \cdot)$ can take. We denote these different distance values by $\delta_1(d, U)$, $\delta_2(d, U)$, ..., $\delta_{\mathrm{span}(d,U)}(d, U)$. We also use $\delta_0(d, U) = 0$. For $t = 0, 1, ..., \mathrm{span}(d, U)$ and alternatives $a, b \in A$, we denote by $N^t_{a|b}(d, U)$ the class of sets of alternatives S that contain alternative a but not alternative b and satisfy $d(U, S) \leq \delta_t(d, U)$.

Definition 4 (majority-concentricity). *A distance metric d is called majority-concentric*[2] *if for*

[2] Majority-concentricity is similar in spirit with a property of distance metrics over rankings with the same name in [8].

every k-sized set of alternatives U, it holds $N^t_{a|b}(d,U) \geq N^t_{b|a}(d,U)$ for every alternatives $a \in U$ and $b \notin U$ and $t = 0, 1, ..., \mathrm{span}(d,U)$.

We are ready to prove our characterization for AV.

Theorem 4. *AV is d-monotone robust if and only if the distance metric d is majority-concentric.*

Proof. Let $\mathcal{M}$ be a d-monotonic noise model for a majority concentric distance metric d. Let U and V be two different sets with k alternatives each. By Lemma 2, in order to show that AV is accurate in the limit for $\mathcal{M}$ (and, consequently, d-monotone robust), it suffices to show that $\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_{\mathrm{AV}}(U,S) - \mathrm{sc}_{\mathrm{AV}}(V,S)] > 0$.

We will need some additional notation. For $t = 0, 1, ..., \mathrm{span}(d,U)$, we denote by $\bar{N}^t(d,U))$ the class of sets of alternatives S that satisfy $d(U,S) = \delta_t(d,U)$. For alternatives $a, b \in A$, we denote $\bar{N}^t_a(d,U))$ the subclass of $\bar{N}^t(d,U)$ consisting of sets of alternatives that include a and by $\bar{N}^t_{a|b}(d,U)$ the subclass of $\bar{N}^t_a(d,U)$ consisting of sets do not contain alternative b.

To simplify notation, we set $s = \mathrm{span}(d,U)$. Also, we drop (d,U) (e.g., we use $N^t_{a|b}$ instead of $N^t_{a|b}(d,U))$ from notation since it is clear from context. We have

$$\begin{aligned}\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_{\mathrm{AV}}(U,S)] &= \sum_{S\subseteq A} \mathrm{sc}_{\mathrm{AV}}(U,S)\cdot \Pr_{\mathcal{M}}[S|U] = \sum_{S\subseteq A} |U\cap S| \cdot \Pr_{\mathcal{M}}[S|U] \\ &= \sum_{a\in U}\sum_{S\subseteq A: a\in S} \Pr_{\mathcal{M}}[S|U] = \sum_{a\in U}\sum_{t=0}^{s}\sum_{S\in\bar{N}^t_a} \Pr_{\mathcal{M}}[S|U]. \end{aligned} \tag{3}$$

Now, observe that the probability $\Pr_{\mathcal{M}}[S|U]$ is the same for all sets $S \in \bar{N}^t$. In the following, we use $p_t = \Pr_{\mathcal{M}}[S|U]$ for all $S \in \bar{N}^t$, for $t = 0, 1, ..., s$. Hence, (3) becomes

$$\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_{\mathrm{AV}}(U,S)] = \sum_{a\in U}\sum_{t=0}^{s} |\bar{N}^t_a|\cdot p_t$$

Similarly, we have

$$\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_{\mathrm{AV}}(V,S)] = \sum_{a\in V}\sum_{t=0}^{s} |\bar{N}^t_a|\cdot p_t,$$

and, by linearity of expectation,

$$\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_{\mathrm{AV}}(U,S) - \mathrm{sc}_{\mathrm{AV}}(V,S)] = \sum_{a\in U\setminus V}\sum_{t=0}^{s} |\bar{N}^t_a|\cdot p_t - \sum_{a\in V\setminus U}\sum_{t=0}^{s} |\bar{N}^t_a|\cdot p_t. \tag{4}$$

Let $\mu : V\setminus U \to U\setminus V$ be a bijection that maps each alternative of $V\setminus U$ to a distinct alternative of $U\setminus V$. Then, (4) becomes

$$\begin{aligned}&\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_{\mathrm{AV}}(U,S) - \mathrm{sc}_{\mathrm{AV}}(V,S)] \\ &= \sum_{a\in U\setminus V}\sum_{t=0}^{s} |\bar{N}^t_a|\cdot p_t - \sum_{a\in U\setminus V}\sum_{t=0}^{s} |\bar{N}^t_{\mu(a)}|\cdot p_t = \sum_{a\in U\setminus V}\sum_{t=0}^{s} \left(|\bar{N}^t_{a|\mu(a)}| - |\bar{N}^t_{\mu(a)|a}|\right)\cdot p_t \\ &= \sum_{a\in U\setminus V} \left(|N^0_{a|\mu(a)}| - |N^0_{\mu(a)|a}|\right)\cdot p_0 + \sum_{a\in U\setminus V}\sum_{t=1}^{s} \left(|N^t_{a|\mu(a)}| - |N^{t-1}_{a|\mu(a)}| - |N^t_{\mu(a)|a}| + |N^{t-1}_{\mu(a)|a}|\right)\cdot p_t\end{aligned}$$

$$= \sum_{a\in U\setminus V}\sum_{t=0}^{s-1}\left(|N^t_{a|\mu(a)}| - |N^t_{\mu(a)|a}|\right)\cdot(p_t - p_{t+1}) + \left(|N^s_{a|\mu(a)}| - |N^s_{\mu(a)|a}|\right)\cdot p_s \qquad (5)$$

$$\geq \sum_{a\in U\setminus V}\left(|N^0_{a|\mu(a)}| - |N^0_{\mu(a)|a}|\right)\cdot(p_0 - p_1) > 0.$$

The third equality follows since $\bar{N}^0_{a|\mu(a)} = N^0_{a|\mu(a)}$, $\bar{N}^0_{\mu(a)|a} = N^0_{\mu(a)|a}$, and $\bar{N}^t_{a|\mu(a)} = N^t_{a|\mu(a)} \setminus N^{t-1}_{a|\mu(a)}$ and $\bar{N}^t_{\mu(a)|a} = N^t_{\mu(a)|a} \setminus N^{t-1}_{\mu(a)|a}$ for $t = 1, ..., s$. The first inequality follows since d is majority concentric and since $p_t > p_{t+1}$ and, thus, all differences in (5) are non-negative. The last inequality follows after observing that since $|N^0_{a|\mu(a)}| = 1$ and $|N^0_{\mu(a)|a}| = 0$ for $a \in U \setminus V$ and since $p_0 > p_1$. This completes the "if" part of the proof.

Let us now consider a non-majority concentric distance metric d that satisfies $N^{t^*}_{a|b}(d, U) < N^{t^*}_{b|a}(d, U)$ for the k-sized set of alternatives U, some alternatives $a \in U$ and $b \notin U$, and some $t^* \in \{1, 2, ..., \mathrm{span}(d, U)\}$. We show the "only if" part of the theorem by constructing a noise model $\mathcal{M}$ that satisfies $\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc_{AV}}(U, S) - \mathrm{sc_{AV}}(V, S)] \leq 0$ for $V = U \setminus \{a\} \cup \{b\}$.

Again, we use $p_t = \Pr_{\mathcal{M}}[S|U]$ for every set of alternatives $S \in \bar{N}^t(d, U)$, $s = \mathrm{span}(d, U)$, and drop (d, U) from notation. We define the model probabilities so that $\tau = p_0 > p_1 > ... > p_{t^*} = \tau - \epsilon$ and $2\epsilon = p_{t^*+1} > ... > p_s = \epsilon$. Notice that such a noise model exists for any arbitrarily small $\epsilon > 0$. Since there are 2^m sets of alternatives and τ is the probability that $\mathcal{M}$ returns the ground truth ranking, it must be $\tau > 1/2^m$. We now apply equality (5). Observe that, since $V = U \setminus \{a\} \cup \{b\}$, $\mu(a) = b$. We obtain

$$\begin{aligned}
&\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc_{AV}}(U, S) - \mathrm{sc_{AV}}(V, S)]\\
&= \sum_{t=0}^{s-1}\left(|N^t_{a|b}| - |N^t_{b|a}|\right)\cdot(p_t - p_{t+1}) + \left(|N^s_{a|b}| - |N^s_{b|a}|\right)\cdot p_s\\
&= \sum_{t=0}^{t^*-1}\left(|N^t_{a|b}| - |N^t_{b|a}|\right)\cdot(p_t - p_{t+1}) + \left(|N^{t^*}_{a|b}| - |N^{t^*}_{b|a}|\right)\cdot(p_{t^*} - p_{t^*+1})\\
&\quad + \sum_{t=t^*+1}^{s-1}\left(|N^t_{a|b}| - |N^t_{b|a}|\right)\cdot(p_t - p_{t+1}) + \left(|N^s_{a|b}| - |N^s_{b|a}|\right)\cdot p_s.
\end{aligned}$$

Now, observe that for $t \neq t^*$, it holds $|N^t_{a|b}| - |N^t_{b|a}| \leq 2^m$ (the total number of sets of alternatives) and $p_t - p_{t+1} \leq \epsilon$. Also, $|N^{t^*}_{a|b}| - |N^{t^*}_{b|a}| \leq -1$ and $p_{t^*} - p_{t^*+1} = \tau - 3\epsilon$. Setting specifically $\epsilon = \frac{1}{s8^m}$, we obtain that

$$\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc_{AV}}(U, S) - \mathrm{sc_{AV}}(V, S)] \leq s2^m\epsilon - (\tau - 3\epsilon) \leq \frac{1}{4^m} - \frac{1}{2^m} + \frac{3}{s\cdot 8^m},$$

which is negative for $m \geq 2$ since $s \geq 1$. The proof of the "only if" part of the theorem now follows by Lemma 2. □

It is tempting to conjecture that AV and MC are the only ABC rules that are monotone robust against all majority concentric distance metrics. However, this is not true as the next example, which uses a different ABC rule, shows.

Example 1. *Let $A = \{a, b, c\}$ and $k = 2$. Consider the majority concentric distance metric d and the ABC rule f that has $f(1, 1) = 1$, $f(2, 2) = 2$, and $f(x, y) = 0$ otherwise. We will show that f is d-monotone robust against any majority concentric distance metric d. Without loss of generality, let us assume that $U = \{a, b\}$ and $V = \{a, c\}$. Observe that the quantity*

$\mathrm{sc}_f(U,S)-\mathrm{sc}_f(V,S)$ *is equal to* 0 *when* $S=\emptyset,\{a\},\{b,c\},\{a,b,c\}$, 1 *when* $S=\{b\}$, -1 *when* $S=\{c\}$, 2 *when* $S=\{a,b\}$, *and* -2 *when* $S=\{a,c\}$. *Hence, for the d-monotonic noise model* $\mathcal{M}$, *we have* $\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_f(U,S)-\mathrm{sc}_f(V,S)]=2p_{ab}-2p_{ac}+p_b-p_c$, *where* p_{ab}, p_{ac}, p_b, *and* p_c *are abbreviations for the probabilities* $\Pr_{\mathcal{M}}[S|U]$ *for* $S=\{a,b\}$, $\{a,c\}$, $\{b\}$, *and* $\{c\}$, *respectively.*

In order to have $N^t_{b|c}\geq N^t_{c|b}$ *for* $t=0,1,...,\mathrm{span}(d,U)$ *as the definition of majority concentricity requires, it must be either* $p_{ab}>p_b,p_c\geq p_{ac}$ *or* $p_{ab}>p_b,p_{ac}\geq p_c$. *In the first case, we have* $\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_f(U,S)-\mathrm{sc}_f(V,S)]=(p_{ab}-p_{ac})+(p_{ab}-p_c)+(p_b-p_{ac})>0$. *In the second case, we have* $\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_f(U,S)-\mathrm{sc}_f(V,S)]=2(p_{ab}-p_{ac})+(p_b-p_c)>0$. *Accuracy in the limit of the ABC rule f for the noise model* $\mathcal{M}$ *follows by Lemma 2.*

5 Robustness of other ABC rules

Our results for other ABC rules (besides MC and AV) involve two classes of distance metrics. We define the first one here.

Definition 5 (natural distance metric)**.** *A distance metric d is called natural if for every three sets* U, V, *and* S *with* $|U|=|V|$ *such that* $|U\cap S|>|V\cap S|$, *it holds that* $d(U,S)\leq d(V,S)$.

The next observation follows easily by the definitions.

Claim 5. *Any natural distance metric is majority-concentric.*

Proof. Let d be a natural distance, U a k-sized set of alternatives, and $a,b\in A$ with $a\in U$ and $b\notin U$. We will show that $|N^t_{a|b}(d,U)|\geq|N^t_{b|a}(d,U)|$ for $t=0,1,...,\mathrm{span}(d,U)$. For $t=0$, this is clearly true since $N^0_{a|b}(d,U)=\{U\}$ and $N^0_{b|a}(d,U)=\emptyset$.

For $t\geq 1$, let $V=U\setminus\{a\}\cup\{b\}$ and μ be any (U,V)-bijection on sets of alternatives. Let $S\in N^t_{b|a}(d,U)$. By the definition of μ, $\mu(S)$ contains alternative a but not b. Also $|U\cap\mu(S)|=|U\cap S|+1$ and, due to naturality of d, $d(U,\mu(S)\leq d(U,S)$. We conclude that $\mu(S)\in N^t_{a|b}(d,U)$. Since μ is a bijection (the sets of $N^t_{b|a}(d,U)$ are mapped to distinct sets in $N^t_{a|b}(d,U)$), we get $|N^t_{a|b}(d,U)|\geq|N^t_{b|a}(d,U)|$, as desired. □

The opposite is not true as the next example illustrates.

Example 2. *Let* $A=\{a,b,c\}$ *and consider the distance metric with* $d(X,Y)=0$ *for every pair of sets with* $X=Y$, $d(X,Y)=1$ *if* $X\cap Y=\emptyset$ *and* $X\cup Y=A$, *and* $d(X,Y)=2$, *otherwise. It can be easily seen that the distance is majority-concentric; it suffices to observe that, within distance* 1 *from any set, each alternative appears in exactly one set. To see that is not natural, consider* $U=\{a,b\}$, $V=\{a,c\}$ *and* $S=\{b\}$. *We have* $|U\cap S|>|V\cap S|$ *but* $d(U,S)=2>1=d(V,S)$.

Lemma 7 below identifies the class of ABC rules that are monotone robust against all natural distance metrics. The condition uses an appropriately defined bijection on sets of alternatives.

Definition 6. *Given two different sets* U *and* V *with* $|U|=|V|$, *a* (U,V)*-bijection* $\mu:2^A\rightarrow 2^A$ *is defined as* $\mu(S)=\{\mu'(a):a\in S\}$, *where* $\mu':A\rightarrow A$ *is such that* $\mu'(a)=a$ *for every alternative* $a\in U\cap V$ *or* $a\notin U\cup V$, $\mu'(a)$ *is a distinct alternative in* $V\setminus U$ *for* $a\in U\setminus V$, *and* $\mu'(a)$ *is a distinct alternative in* $U\setminus V$ *for* $a\in V\setminus U$.

It is easy to see that a (U,V)-bijection μ has the following properties.

Claim 6. *Let* $U,V\subseteq A$ *with* $|U|=|V|$ *and let* μ *be a* (U,V)*-bijection. For every* $S\subseteq A$, *it holds* $|S|=|\mu(S)|$, $|U\cap S|=|V\cap\mu(S)|$, *and* $|U\cap\mu(S)|=|V\cap S|$.

Lemma 7. *An ABC rule is d-monotone robust against a natural distance metric d if and only if for every two different sets of alternatives $U, V \subseteq A$ with $|U| = |V| = k$ there exists a (U,V)-bijection μ on sets of alternatives and a set $S \subseteq A$ with $\mathrm{sc}_f(U,S) > \mathrm{sc}_f(V,S)$ and $d(U,S) < d(U,\mu(S))$.*

Proof. Let U and V be two different sets with k alternatives each. Let $\mathcal{S}_+$, $\mathcal{S}_-$, and $\mathcal{S}_0$ be the classes of sets of alternatives S with $|U \cap S| > |V \cap S|$, $|U \cap S| < |V \cap S|$, and $|U \cap S| = |V \cap S|$, respectively. Using this notation, we have

$$\begin{aligned}
&\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_f(U,S) - \mathrm{sc}_f(V,S)] = \sum_{S\subseteq A} (\mathrm{sc}_f(U,S) - \mathrm{sc}_f(V,S)) \cdot \mathrm{Pr}_{\mathcal{M}}[S|U] \\
&= \sum_{S\in\mathcal{S}_+} (\mathrm{sc}_f(U,S) - \mathrm{sc}_f(V,S)) \cdot \mathrm{Pr}_{\mathcal{M}}[S|U] + \sum_{S\in\mathcal{S}_0} (\mathrm{sc}_f(U,S) - \mathrm{sc}_f(V,S)) \cdot \mathrm{Pr}_{\mathcal{M}}[S|U] \\
&\quad + \sum_{S\in\mathcal{S}_-} (\mathrm{sc}_f(U,S) - \mathrm{sc}_f(V,S)) \cdot \mathrm{Pr}_{\mathcal{M}}[S|U] \qquad (6)
\end{aligned}$$

We will now transform the third sum in the RHS of (6) to one running over the sets of $\mathcal{S}_+$ like the first sum.

Let μ be a (U,V)-bijection on sets of alternatives; by Claim 6, μ maps every set of $\mathcal{S}_-$ to a set of $\mathcal{S}_+$ and vice-versa. Hence, instead of enumerating sets of $\mathcal{S}_-$, we could enumerate sets of $\mathcal{S}_+$ and apply the bijection μ on them. The third sum in the RHS of (6) then becomes

$$\begin{aligned}
\sum_{S\in\mathcal{S}_-} (\mathrm{sc}_f(U,S) - \mathrm{sc}_f(V,S)) \cdot \mathrm{Pr}_{\mathcal{M}}[S|U] &= \sum_{S\in\mathcal{S}_+} (\mathrm{sc}_f(U,\mu(S)) - \mathrm{sc}_f(V,\mu(S))) \cdot \mathrm{Pr}_{\mathcal{M}}[\mu(S)|U] \\
&= \sum_{S\in\mathcal{S}_+} (\mathrm{sc}_f(V,S) - \mathrm{sc}_f(U,S)) \cdot \mathrm{Pr}_{\mathcal{M}}[\mu(S)|U] \qquad (7)
\end{aligned}$$

The second equality follows since, by Claim 6, $\mathrm{sc}_f(U,\mu(S)) = f(|U \cap \mu(S)|, |\mu(S)|) = \mathrm{sc}_f(|V \cap S|, |S|) = \mathrm{sc}_f(V,S)$ and, similarly, $\mathrm{sc}_f(V,\mu(S)) = \mathrm{sc}_f(U,S)$.

Now observe that $\mathrm{sc}_f(U,S) = \mathrm{sc}_f(V,S)$ when $S \in \mathcal{S}_0$. Hence, the second sum in the RHS of (6) is equal to 0. By combining (6) and (7), we get

$$\begin{aligned}
&\mathbb{E}_{S\sim\mathcal{M}(U)}[\mathrm{sc}_f(U,S) - \mathrm{sc}_f(V,S)] \\
&= \sum_{S\in\mathcal{S}_+} (\mathrm{sc}_f(U,S) - \mathrm{sc}_f(V,S)) \cdot (\mathrm{Pr}_{\mathcal{M}}[S|U] - \mathrm{Pr}_{\mathcal{M}}[\mu(S)|U]). \qquad (8)
\end{aligned}$$

Now observe that the RHS of (8) is always non-negative. This is due to the fact that $S \in \mathcal{S}_+$ which implies that $\mathrm{sc}_f(U,S) = f(|U \cap S|, |S|) \geq f(|V \cap S|, |S|) = \mathrm{sc}_f(V,S)$ since f is non-decreasing in its first argument and $d(U,S) \leq d(U,\mu(S))$ (and, consequently, $\mathrm{Pr}_{\mathcal{M}}[S|U] \geq \mathrm{Pr}_{\mathcal{M}}[\mu(S)|U]$) since d is natural.

Clearly, the RHS of (8) is strictly positive if and only if there exists a set $S \in \mathcal{S}_+$ that further satisfies $d(U,S) < d(V,S)$ (and, consequently, $\mathrm{Pr}_{\mathcal{M}}[S|U] > \mathrm{Pr}_{\mathcal{M}}[\mu(S)|U]$) and $\mathrm{sc}_f(U,S) > \mathrm{sc}_f(V,S)$ (notice that sets that do not belong to $\mathcal{S}_+$ cannot satisfy these conditions). The theorem then follows by Lemma 2. □

We now present two applications of Lemma 7. The first one involves the natural distance metrics.

Theorem 8. *An ABC rule f is monotone robust against any natural distance metric if and only if $f(k,k) > f(k-1,k)$.*

Proof. For the proof of the "if" part, consider any pair of different k-sized sets alternatives U and V and any (U, V)-bijection μ. For $S = U$, it holds $d(U,U) < d(U,\mu(U))$ and $\mathrm{sc}_f(U,U) > \mathrm{sc}_f(V,U)$ by the definition of f.

For the proof of the "only if" part, assume that $f(k,k) = f(k-1,k)$ and consider the natural distance metric d with $d(X,Y) = 0$ if $X = Y$ and $d(X,Y) = 1$ otherwise. Let $U, V \subseteq A$ be different sets with k alternatives each such that $U \cap V \neq \emptyset$. Observe that the only set $S \subseteq A$ that satisfies $d(U,S) < d(V,S)$ is U itself. But then (for $S = U$), it holds $\mathrm{sc}_f(U,S) = \mathrm{sc}_f(V,S)$ and the condition of Lemma 7 does not hold. Hence, f is not monotone robust against d. $\square$

Notice that most popular ABC rules from Section 2 satisfy the condition of Theorem 8. CC is an exception; Theorem 8 implies that CC is not monotone robust for some natural distance metric.

Our second application of Lemma 7 involves all non-trivial ABC rules and an important subclass of natural distances.

Definition 7 (similarity distance metric)**.** *A natural distance metric d is a similarity distance metric if for every three sets U, V, and S with $|U| = |V|$ such that $|U \cap S| > |V \cap S|$, it holds that $d(U,S) < d(V,S)$.*

Theorem 9. *Any non-trivial ABC rule is monotone robust against any similarity distance metric.*

Proof. We apply Lemma 7 assuming a non-trivial ABC rule f and a similarity distance metric d. Non-triviality of f implies that for every two different sets U and V with k alternatives each, there is a set S such that $\mathrm{sc}_f(U,S) > \mathrm{sc}_f(V,S)$. This immediately yields $|U \cap S| > |V \cap S|$, which implies that $d(U,S) < d(V,S)$ since d is a similarity distance. $\square$

We can easily show that the four distance metrics set difference, Jaccard, Zelinka, and Bunke-Shearer that we defined in Section 2 are all similarity distance metrics. Using this observation and Theorem 9, we obtain the next statement.

Corollary 10. *Any non-trivial ABC rule is monotone robust against the set difference, Jaccard, Zelinka, and Bunke-Shearer distance metrics.*

6 Epilogue

We believe that our approach complements nicely the axiomatic and quantitative analysis of approval-based multiwinner voting. The current paper leaves many interesting open problems. Besides identifying ABC rules that are at least as robust as AV, applying our framework to non-ABC rules deserves investigation. Beyond assessing the effects of noise in the limit, studying the sample complexity of approval-based multiwinner voting is important. This will require the design of concrete noise models like the $\mathcal{M}_p$ model that we presented in Section 2. In particular, models that simulate user behaviour in crowdsourcing platforms will be useful for evaluating approval-based voting in such environments. Even though the $\mathcal{M}_p$ model is very simple, we expect that implementation issues will emerge for more elaborate noise models. Similar issues in the implementation of the Mallows [18] ranking model have triggered much non-trivial work; see, e.g., [12].

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