Temporal Network Optimization Subject to Connectivity Constraints

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joint work with
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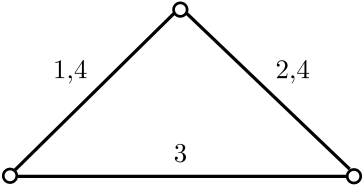
Temporal Networks

Definition (Temporal Graph)

Let $G = (V, E)$ be a (di)graph and $\lambda : E \to 2^{\mathbb{N}}$ be a labeling of G. Then $\lambda(G)$ is the temporal graph (or dynamic graph) of G w.r.t. λ.
Furthermore, G is the underlying graph of $\lambda(G)$.

- Loosely speaking a network that changes with time
- Labels indicate availability times of edges



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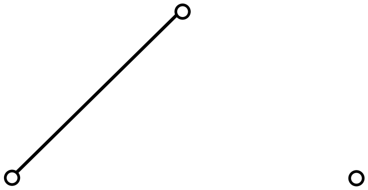

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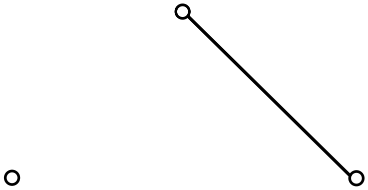

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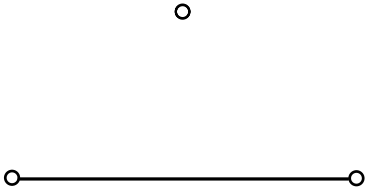

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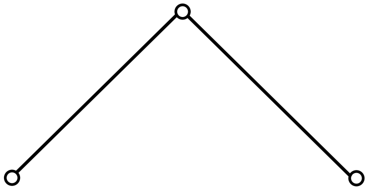
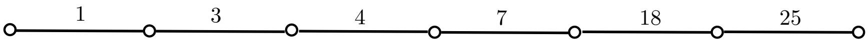
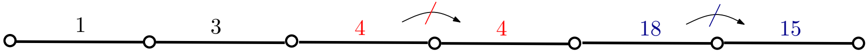


- Paths with strictly increasing labels
- a.k.a. journeys
- A journey:



Time-Respecting Paths

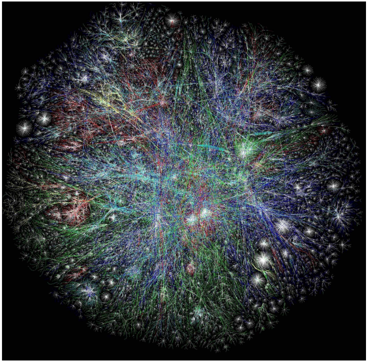
- Paths with strictly increasing labels
- a.k.a. journeys
- A non-journey:



Motivation

A great variety of systems are dynamic:

- Modern communication networks: inherently dynamic, dynamicity may be of high rate
  - mobile ad hoc, sensor, peer-to-peer, opportunistic, and delay-tolerant networks
- Social networks: social relationships between individuals change, existing individuals leave, new individuals enter



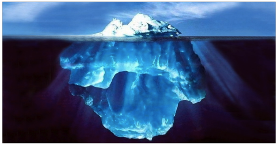
- Transportation networks: transportation units change their positions in the network as time passes
- Physical systems: e.g. systems of interacting particles

State of the Art

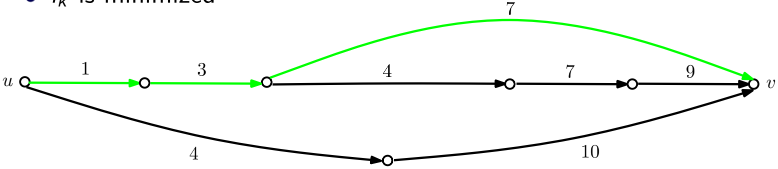
- Traditional communication networks: topology modifications are rare
- The structural and algorithmic properties of temporal graphs are not well understood yet
- Single-label temporal graphs
  - The max-flow min-cut theorem holds with unit capacities [Berman, '96]
  - Menger's theorem is violated [Kempe, Kleinberg, Kumar, STOC, '00]
- Continuous availabilities (intervals)
  - natural model but different techniques (e.g. journey problems [Xuan *et al.*, IJFCS, '03], dynamic flows [Fleischer, Tardos, Op. Res. Let., '98])
- Distributed Computing on Dynamic Networks
  - Worst-case dynamicity [Kuhn, Lynch, Oshman, STOC, '10], [Michail, Chatzigiannakis, Spirakis, JPDC, '13]
  - Population Protocols (interacting automata) [Angluin *et al.*, Distr. Comp., '06], [Michail, Chatzigiannakis, Spirakis, Book, '11]
  - Randomly Dynamic Networks [Clementi *et al.*, PODC, '08]

Overview-Contribution

- We give efficient algorithms for shortest journeys
- We state a temporal analogue of Menger's theorem and prove it valid for arbitrary temporal graphs
- We define cost minimization parameters for temporal network design
  - temporality, temporal cost, and age
  - satisfy some connectivity property: all paths and all reachabilities
  - We provide upper and lower bounds for basic graph families, e.g. rings, DAGs, trees, and a trade-off between temporality and age
  - We give a generic method for lower-bounding the temporality
  - **APX**-hardness result for temporal cost and an approximation algorithm
- Just the tip of the iceberg...



- Foremost (u, v)-journey $(e_1, l_1), (e_2, l_2), \ldots, (e_k, l_k)$ from time t
  - $l_1 \geq t$ and
  - l_k is minimized



Theorem

Let $\lambda(G)$ be a temporal graph, $s \in V$ be a source node, and t_{start} a time s.t. $\lambda_{\min} \leq t_{start} \leq \lambda_{\max}$. We provide an algorithm that correctly computes for all $v \in V \backslash \{s\}$ a foremost (s, v)-journey from time t_{start}. The running time of the algorithm is $O(n\alpha^3(\lambda) + |\lambda|)$.

- $\alpha(\lambda) = \lambda_{\max} - \lambda_{\min} + 1$: the age of a temporal graph

Journey Problems

- Weighted temporal graph: In addition to λ a positive weight $w(e)$ is assigned to every $e \in E$
- Shortest Journey: Minimizes the sum of the weights of its edges
- Let $\lambda(G)$ be a weighted single-label temporal graph (i.e. $|\lambda(e)| = 1$ for all $e \in E$)

Theorem

For any two nodes $s, v \in V$, we can compute a shortest journey between s and v in $\lambda(G)$ (or report that no such journey exists) in $O(m \log m + \sum_{u \in V} \delta_u^2) = O(n^3)$ time.

- δ_u is the degree of node u, $m = |E|$

Menger's Analogue for Temporal Graphs

Theorem (Menger's theorem)

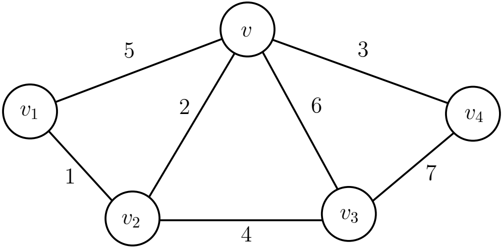
The maximum number of node-disjoint s-v paths is equal to the minimum number of nodes needed to separate s from v.

- Does it carry over to temporal graphs?
- Previous known result was negative

Theorem (Kempe, Kleinberg, Kumar, STOC '00)

There is no analogue of Menger's theorem, at least in its original formulation, for arbitrary single-label temporal networks.

- A violation of Menger's theorem



- There are no two disjoint time-respecting paths from v_1 to v_4 but
- After deleting any one node (other than v_1 or v_4) there still remains a time-respecting v_1-v_4 path

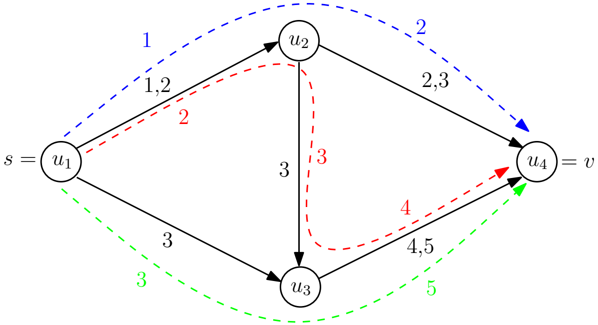
- We give a positive result
  - An analogue of Menger's theorem valid for all temporal networks
- Two journeys are out-disjoint if they never leave from the same node at the same time
- Remove departure time t from node u:
  - for all edges (u, w), remove label t from (u, w) (if it exists)

Theorem

Take any temporal graph $\lambda(G)$ with two distinguished nodes s and v. The maximum number of out-disjoint journeys from s to v is equal to the minimum number of node departure times needed to separate s from v.

Menger's Analogue: An Example

- 3 out-disjoint journeys from s to v



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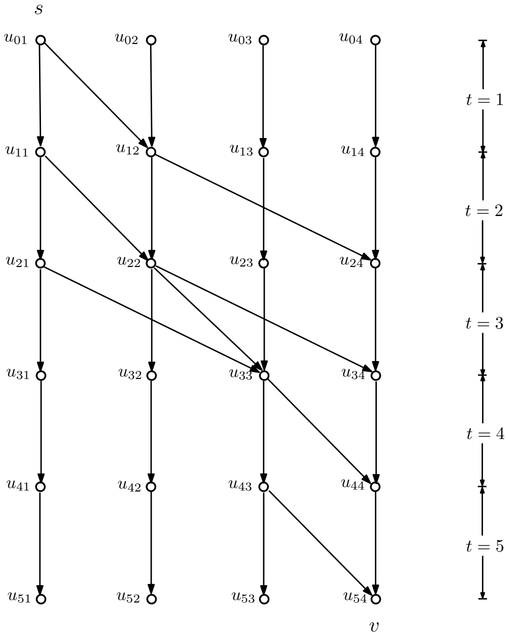

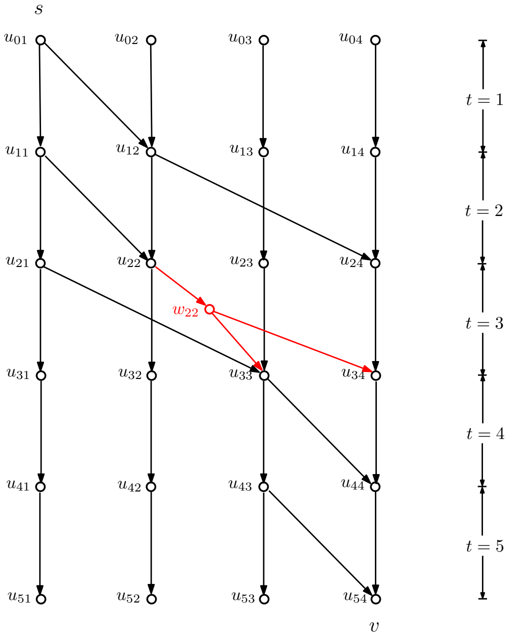
s
u_{01}
u_{02}
u_{03}
u_{04}
u_{11}
u_{12}
u_{13}
u_{14}
u_{21}
u_{22}
u_{23}
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u_{31}
u_{32}
u_{33}
u_{34}
u_{41}
u_{42}
u_{43}
u_{44}
u_{51}
u_{52}
u_{53}
u_{54}
v
$t = 1$
$t = 2$
$t = 3$
$t = 4$
$t = 5$

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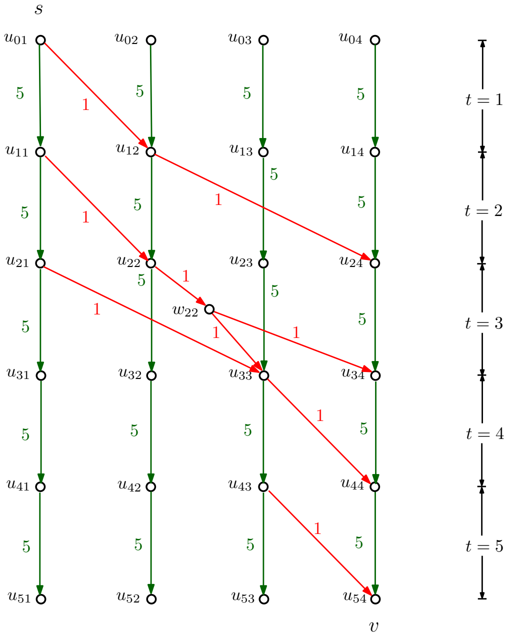

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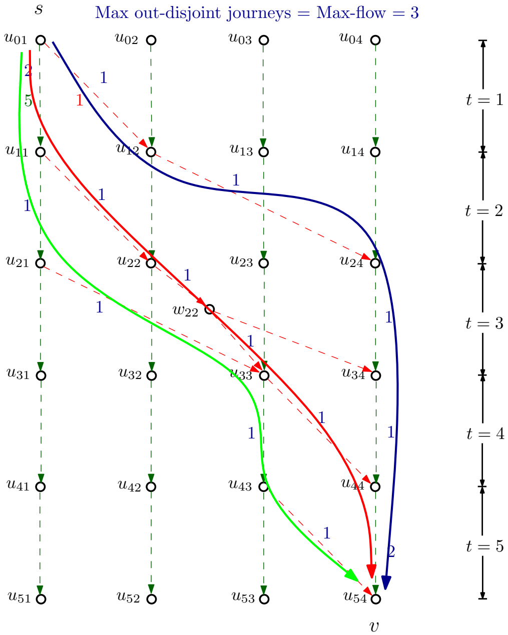


Menger's Analogue: Distributed Token Gathering

- Distributed Model
  - $\lambda(G, t)$ is connected at all times $t \in \mathbb{N}$
  - $k \leq n$ tokens assigned to some given source nodes $S \subseteq V$
  - In each (discrete, synchronous) round i, each node broadcasts a single token to all its current neighbors (i.e. those defined by $E(i)$)

Lemma (Dutta *et al.*, SODA, '13)

All the tokens can be sent to any given v in $O(n)$ rounds.

- We substantially simplify the proof via Menger's Temporal Analogue.
- Given a mapping $N : S \to \mathbb{N}_{\geq 1}$ so that $\sum_{s \in S} N(s) = k$, we prove:

Lemma

Let the age be $\alpha(\lambda) = n + k$. There are at least k out-disjoint journeys from S to any given v such that $N(s_i)$ journeys leave from each source node s_i.

Cost Minimization Parameters

- (Di)graph $G = (V, E)$, $\alpha_{\max} \in \mathbb{N}$: upper bound on the age, Connectivity property $\mathcal{P}$

Definition (Temporality)

The temporality of $(G, \mathcal{P}, \alpha_{\max})$ is

$$\tau(G, \mathcal{P}, \alpha_{\max}) = \min_{\lambda \in \mathcal{P} \cap \mathcal{L}_{G,\alpha_{\max}}} \max_{e \in E} |\lambda(e)|$$

- i.e. minimize the maximum number of labels of an edge while satisfying $\mathcal{P}$ and having age at most $\alpha_{\max}$

Definition (Temporal Cost)

The temporal cost of $(G, \mathcal{P}, \alpha_{\max})$ is

$$\kappa(G, \mathcal{P}, \alpha_{\max}) = \min_{\lambda \in \mathcal{P} \cap \mathcal{L}_{G,\alpha_{\max}}} \sum_{e \in E} |\lambda(e)|$$

- i.e. minimize the total number of labels used

Cost Minimization Parameters

- Similarly we define the age optimization criterion
- $\tau_{\max} \in \mathbb{N}$: upper bound on the temporality

Definition (Age)

The age of $(G, \mathcal{P}, \tau_{\max})$ is

$$\alpha(G, \mathcal{P}, \tau_{\max}) = \min_{\lambda \in \mathcal{P} \cap \mathcal{L}_{G, \tau_{\max}}} \alpha(\lambda)$$

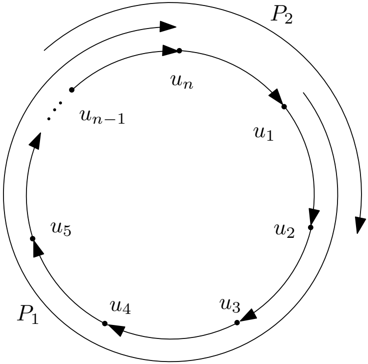
- i.e. minimize the age while satisfying $\mathcal{P}$ and having temporality at most $\tau_{\max}$
- Minimizing such parameters is crucial for many real networks
  - Establishing and maintaining a connection does not come for free
  - e.g. in WSNs cost of edges is directly related to: power consumption of keeping nodes awake, broadcasting, listening, resolving collisions

Connectivity Properties

- λ preserves path P if it gives a journey on P
- We investigate the following connectivity properties
  - all-paths(G)$= \{\lambda \in \mathcal{L}_G$: for all simple paths P of G, λ preserves $P\}$
  - reach(G)$= \{\lambda \in \mathcal{L}_G$: for all $u, v \in V$ where v is reachable from u in G, λ preserves at least one simple path from u to $v\}$.
- Example
  - Given: directed ring $R = u_1, u_2, \ldots, u_n$
  - Problem: determine $\tau(R, all\ paths)$, i.e. the temporality of the ring subject to the all paths property (no constraint on the age here)
  - i.e. find a labeling λ that (i) preserves every simple path of the ring and (ii) at the same time minimizes the maximum number of labels of an edge

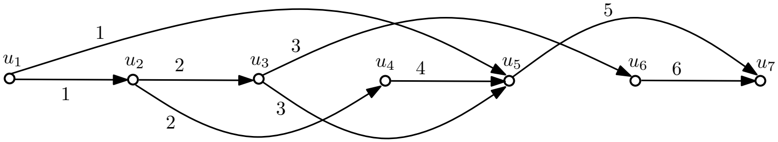
Ring Temporality Subject to All Paths

- Increasing labels on P_1 $\Rightarrow$ decreasing labels on (u_{n-1}, u_n) and (u_1, u_2)
- But P_2 uses first (u_{n-1}, u_n) and then (u_1, u_2) thus requires an increasing pair of labels on these edges
- To preserve both P_1, P_2 must use 2 labels on at least one of these two edges $\Rightarrow \tau(R, \textit{all paths}) \geq 2$



- The labeling that assigns to each edge (u_i, u_{i+1}) the labels $\{i, n+i\}$ preserves all simple paths, i.e. $\tau(R, \textit{all paths}) \leq 2$
- Conclusion: $\tau(R, \textit{all paths}) = 2$

Preserving All Paths of a DAG



Proposition

If G is a DAG then $\tau(G, \text{all paths}) = 1$.

Proof.

- Take a topological sort $u_1, u_2, \ldots, u_n$ of G
- Give to every edge (u_i, u_j), where $i < j$, label i

□

- It is sufficient to understand how $\tau(G, reach)$, behaves on strongly connected digraphs
- $\mathcal{C}(G)$: the set of all strongly connected components of a digraph G

Lemma

$\tau(G, reach) \leq \max_{C \in \mathcal{C}(G)} \tau(C, reach)$ *for every digraph* G*.*

- Using this we prove that

Theorem (Generic Upper Bound)

$\tau(G, reach) \leq 2$ *for all digraphs* G*.*

- i.e. we can preserve all reachabilities of any digraph by using at most 2 labels on every edge

Restricting the Age

Theorem

If T is an undirected tree then $\tau(T, all\ paths, d(T)) \leq 2$.

- the age above is restricted to be at most the diameter $d(T)$ of T

Theorem (Age-Temporality Trade-off)

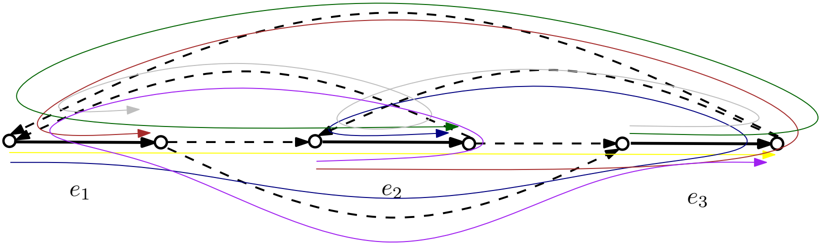
If G is a directed ring and $\alpha = (n-1) + k$, where $1 \leq k \leq n-1$, then

$$\tau(G, all\ paths, \alpha) = \Theta(n/k)$$

- *In particular,* $\lfloor \frac{n-1}{k+1} \rfloor + 1 \leq \tau(G, all\ paths, \alpha) \leq \lceil \frac{n}{k+1} \rceil + 1$
- *Moreover,* $\tau(G, all\ paths, n-1) = n-1$ *(i.e. when $k = 0$)*

Definition (Edge-kernel)

$K = \{e_1, e_2, \ldots, e_k\} \subseteq E(G)$ is an edge-kernel of G if for every permutation $\pi = (e_{i_1}, e_{i_2}, \ldots, e_{i_k})$ of K there is a simple path of G that visits all edges of K in the ordering defined by π.



Generic Method for Lower Bounding Temporality

Theorem (Edge-kernel Lower Bound)

If a digraph G contains an edge-kernel of size k then $\tau(G, \text{all paths}) \geq k$.

Proof.

- $K = \{e_1, e_2, \ldots, e_k\}$: an edge-kernel of size k
- On every e_i sort the labels in an ascending order. $\lambda_l(e)$: the lth smallest label of edge e, e.g. $\lambda(e) = \{1, 3, 7\} \Rightarrow \lambda_1(e) = 1$, $\lambda_2(e) = 3$, $\lambda_3(e) = 7$
- Construct a permutation $\pi = (e_{j_1}, e_{j_2}, \ldots, e_{j_k})$ of K. e_{j_1}: edge with max λ_1, e_{j_2}: edge with max λ_2 between the remaining edges, ...
- Observe that π satisfies $\lambda_i(e_{j_i}) \geq \lambda_i(e_{j_{i+1}})$ for all $1 \leq i \leq k-1$
- π cannot use the labels $\lambda_1, \ldots, \lambda_{i-1}$ at edge e_{j_i} thus at edge e_{j_k} it can use none of the $k-1$ available labels $\Rightarrow$ needs a kth label $\square$

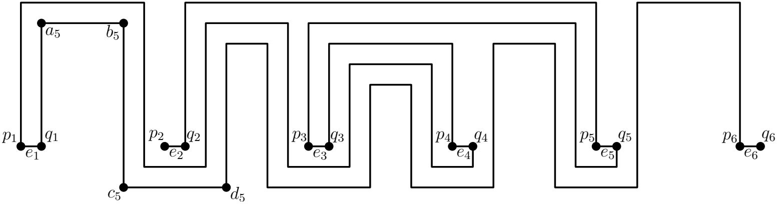
Applying the Edge-kernel Lower Bound

Lemma

If G is a complete digraph of order n then it has an edge-kernel of size $\lfloor n/2 \rfloor$.

Lemma

There exist planar graphs having edge-kernels of size $\Omega(n^{\frac{1}{3}})$.



Hardness of Approximating the Cost

- Max-XOR: given a 2-CNF formula ϕ, max number of clauses of ϕ simultaneously XOR-satisfied in a truth assignment
- Max-XOR(k): every literal appears in at most k clauses of ϕ

Lemma

*The Max-XOR(3) problem is **APX**-hard.*

Theorem

There exists a truth assignment of ϕ XOR-satisfying at least k clauses iff $\kappa(G_\phi, reach, d(G_\phi)) \leq 39n - 4m - 2k$.

Theorem (Hardness of Approximating the Temporal Cost)

*Computing $\kappa(G, reach, d(G))$ is **APX**-hard, even when the maximum length of a directed cycle in G is 2 (i.e. very close to a DAG).*

Approximating the Cost

- $r(u) = |\{v \in V : v \text{ is reachable from } u\}|$
- $r(G) = \sum_{u \in V} r(u)$: total number of reachabilities in G

Theorem

We provide an $\frac{r(G)}{n-1}$-factor approximation algorithm for computing $\kappa(G, reach, d(G))$ on any weakly connected digraph G.

Proof.

- $\text{OPT} \geq n - 1$
- Consider the following algorithm producing a labeling λ:
  - For all $u \in V$, compute a BFS out-tree T_u rooted at u
  - For all T_u, give to each edge at distance i from the root label i
- Maximum label used by λ is $d(G)$ and
- $\text{ALG} = |\lambda| = r(G)$: for each u, we label precisely $r(u)$ edges in T_u $\square$

Research Directions: To name a few...

- Still many interesting graph families to be investigated like regular or bounded-degree graphs
- Are there are other structural properties of G that cause a growth of temporality? (apart from edge-kernels)
- Other natural connectivity properties subject to which optimization is to be performed
  - e.g. preserve a shortest path between every reachable pair
  - depart from paths and require the preservation of more complex subgraphs
- Set the optimization criterion to be the age of λ
  - $\alpha(G, \text{all paths})$ is **NP**-hard (reduction from HAMPATH)
  - 2-factor approximation algorithm for $\alpha(G, \textit{reach}, 2)$

Research Directions: To name a few...

- Great room for approximation and randomized algorithms for all combinations of optimization parameters and connectivity constraints
- Polynomial-time algorithms for specific “easy to handle” graph families
- Consider periodic or probabilistic models of temporal graphs
- Our results are a first step towards answering the following fundamental question:

  To what extent can algorithmic and structural results of graph theory be carried over to temporal graphs?

Thank You!