Limitations of deterministic auction design for correlated bidders

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Abstract. The seminal work of Myerson (Mathematics of OR 81) characterizes incentive-compatible single-item auctions among bidders with independent valuations. In this setting, relatively simple deterministic auction mechanisms achieve revenue optimality. When bidders have correlated valuations, designing the revenue-optimal deterministic auction is a computationally demanding problem; indeed, Papadimitriou and Pierrakos (STOC 11) proved that it is APX-hard, obtaining an explicit inapproximability factor of 99.95%. In the current paper, we strengthen this inapproximability factor to $57/58 \approx 98.3\%$. Our proof is based on a gap-preserving reduction from the problem of maximizing the number of satisfied linear equations in an over-determined system of linear equations modulo 2 and uses the classical inapproximability result of Håstad (J. ACM 01). We furthermore show that the gap between the revenue of deterministic and randomized auctions can be as low as $13/14 \approx 92.9\%$, improving an explicit gap of $947/948 \approx 99.9\%$ by Dobzinski, Fu, and Kleinberg (STOC 11).

1 Introduction

In the classical model of Auction Theory [10], a seller auctions off an item to n bidders with valuations for the item drawn independently from known but not necessarily identical probability distributions. Myerson's seminal work [14] gives an elegant characterization of revenue-maximizing auctions in this setting. Optimal revenue is achieved by simple deterministic auctions that are defined using succinct information about the known probability distributions. In contrast, the case of bidders with correlated valuations has been a mystery; in spite of the vast related literature in Economics and (more recently) in Computer Science, no such general characterization result has been presented so far. Due to their simplicity and amenability to implement in practice, deterministic auctions are of particular importance. Recently, Papadimitriou and Pierrakos [16] provided an explanation — from the computational complexity point of view — for this lack of results, by proving that the problem of designing the optimal deterministic auction given the explicit description of the joint probability distribution (with finite support) is an APX-hard problem. Furthermore, Dobzinski et al. [7] provided a separation between randomized (truthful in expectation) and deterministic truthful auctions; there are (single-item) settings in which randomized auctions may extract strictly more revenue than any deterministic auction. Both results hold even when only three bidders participate in the auction. In this paper, we strengthen both results.

Existing approaches on single-item auctions with correlated bidders fall into three different categories. A first approach that has been mostly followed by economists (e.g, see [13, 11, 12, 3]) assumes that each bidder has her own valuation function that depends on a shared random variable; this model is usually referred to as the *interdependent valuations model*. In a second approach, the support of the joint probability distribution is extremely large (exponentially larger than the number of players or even infinite) and an auction mechanism can obtain information about the distribution through queries. The related literature focuses on the design of auctions that use only a polynomial (in terms of the number of bidders) number of queries. In the third one, the joint probability has finite support and the related work seeks for auctions that are defined in time polynomial in terms of the support size and the number of players. The last two models are known as the *query model* and the *explicit model*, respectively. Among these models, the explicit one allows us to view the design of the revenue-optimal (deterministic) auction as a standard optimization problem. The auction has to define the bidder that gets the item and the payment of this bidder to the auctioneer for every valuation vector of the support of the joint probability distribution. Both the allocation and the payments should be defined in such a way that no bidder has an incentive to misreport her true valuation; this constraint is commonly known as *incentive compatibility*. The objective is to maximize the expected revenue of the auctioneer over all valuation vectors.

Since our purpose is to explore the limitations of deterministic auctions, we focus on the explicit model and the case of three bidders. Following [16], we refer to the optimization problem mentioned above (when restricted to three bidders) as 3OptimalAuctionDesign. The inapproximability bound presented by Papadimitriou and Pierrakos [16] is marginally smaller than 1, namely 1999/2000. It is achieved by a gap-preserving reduction from a structured maximum satisfiability problem called CatSat. This problem has an inapproximability of 79/80; hence, the gap obtained for 3OptimalAuctionDesign is even closer to 1. We present a different reduction from the classical Max-3-Lin(2) problem of approximating the number of satisfied linear equations in an over-determined system of linear equations modulo 2 (with three binary variables per equation). Our proof uses the seminal 1/2-inapproximability result of Håstad [9] and yields a significantly improved inapproximability bound of $57/58 \approx 98.3\%$ for 3OptimalAuctionDesign. Furthermore, we demonstrate a rather significant revenue gap between deterministic truthful mechanisms and randomized auctions that are truthful in expectation; the revenue of any deterministic auction can be at most $13/14 \approx 92.9\%$ of the optimal randomized one. This result improves the previously known explicit bound of $947/948 \approx 99.9\%$ of Dobzinski et al. [7]. Our construction is considerably simpler than the one in [7].

Related work. Extending Myerson's work, Crémer and McLean [4, 5] characterize the information structure that guarantees the auctioneer full surplus, under several settings with correlated valuations. They consider *interim* individual rationality which allows players to have negative utility for some valuation vectors of the joint distribution. In contrast, our work (as well as the recent related work) focuses on *ex post* individual rationality, a design requirement that does not allow such situations. Ronen [17] and Ronen and Saberi [18] consider single-item optimal auctions in the query model. They design auctions that use queries of the following form: given the valuations of a set of players, which is the conditional distribution of the remaining ones? The 1-lookahead auction in [17] yields at least half the optimal revenue to the seller. Essentially, the auction ignores the $n-1$ lowest bids and offers the item to the remaining bidder at the price that maximizes the revenue considering the distribution of valuations of that bidder conditioned on the valuations of everybody else. Ronen and Saberi [18] present several impossibility results for auctions of particular type. For example, they prove that no ascending auction can approximate the optimal revenue to a factor greater than 3/4.

Dobzinski et al. [7] consider k-lookahead auctions (a natural extension of 1-lookahead) and show that a 2/3-approximation of the optimal revenue can be achieved by randomized auctions in the query model. For the explicit model, they show that the optimal randomized auction can be computed by linear programming while the deterministic 2-lookahead auction achieves a 3/5-approximation of revenue. Their positive results have been strengthened by Chen et al. [2] to approximation factors of 0.731 and 0.622, respectively. Both [7] and [16] prove that revenue-optimal auction design can be solved in polynomial time in the 2-bidder case. The 2-bidder case has also been considered in [8] and [6]. In particular, Diakonikolas et al. [6] study the tradeoff between efficiency and revenue in deterministic truthful auctions and prove that any point of the Pareto curve can be approximated with arbitrary precision.

Roadmap. The rest of the paper is structured as follows. We begin with preliminary definitions in Section 2. The reduction and the proof of the inapproximability of 3OptimalAuctionDesign are presented in Section 3. The revenue-gap construction is presented in Section 4. Due to lack of space, several details in the constructions and omitted proofs are presented in Appendix.

2 Preliminaries

We study the setting in which an item is auctioned off among players (or bidders) with correlated valuations and quasi-linear utilities. Players draw their valuations from a joint probability distribution $\mathcal{D}$ over valuation vectors. A single-item auction mechanism is defined by an allocation and a payment function. For every vector of bids that are submitted by the players to the auctioneer, the allocation defines who among the players (if any) gets the item and the payment function decides the payment of the winning player to the auctioneer. Incentive compatibility, individual rationality, and the no-positive-transfers property are classically considered as important desirata for auction mechanisms [10, 15]. Incentive compatibility requires that truth-telling maximizes players' utility (i.e., valuation minus payment). Hence, each player always submits as bid her actual valuation for the item. Individual rationality encourages players to participate. In particular, we consider *ex post* individual rationality that requires that players always have non-negative utility. We also require that the players never receive payments from the auctioneer (no positive transfer). An obvious objective for auction mechanisms is the maximization of the expected revenue, i.e., the expectation over all valuation vectors of the payment received by the auctioneer.

The recent work on revenue-optimal deterministic auction mechanism (e.g., see [16]) restricts the search space to *monotone* allocations and *threshold* payment functions. An allocation is monotone if when the item is allocated to some player i for some bidding vector b, player i is allocated the item when the bid vector is (b', b_{-i}) with $b' > b_i$. The notation (b', b_{-i}) denotes the bidding vector where player i bids b' and the remaining players keep their bids as in b. A threshold payment for a winning player i is then defined as the minimum bid b'' so that player i gets the item for the bidding vector (b'', b_{-i}).

We consider auctions with three players and assume that $\mathcal{D}$ is defined by a set S of points in $\mathbb{R}^3$ and associated weights with these points. The weight of a point indicates the probability that the corresponding valuation vector is realized. We refer to the three players as player x, y, and z; clearly, a point of S implies a valuation vector where players x, y, and z have as valuation the x-, y-, and z-coordinate of the point. Naturally, we will refer to allocations of points to players. In the following, we say that two points are *x-aligned* (resp., y-aligned, z-aligned) if they have the same y- and z-coordinates (resp., x- and z-coordinates, x- and y-coordinates). Monotonicity implies that if a point p is allocated to a player (say, x), then all points that are x-aligned with p and have higher x-coordinate are *enforced* to be allocated to player x as well (similarly for the other players). The payment associated with a point p that is allocated to player x is then the lowest x-coordinate of the x-aligned points with p that are allocated to x (similarly for the other players). So, we can state the problem of designing the optimal deterministic auction mechanism as follows:

3OptimalAuctionDesign: Given a finite set of points $S \subset \mathbb{R}^3$ and associated weights, compute a monotone allocation of the points of S to players x, y, and z so that the weighted sum of the implied threshold payments (expected revenue) is maximized.

Randomized allocations can allocate fractions of points to players under the restriction that the total allocation fraction for a point is at most 1. Fractions correspond to allocation probabilities. In randomized auction mechanisms that are truthful-in-expectation, the allocation is monotone in the sense that the allocation probabilities to player x (similarly for the other players) are non-decreasing in the terms of the x-coordinate of x-aligned points. The payment when player x gets the item at point p depends on the x-coordinate of the points that are x-aligned to p and have lower x-coordinates and their allocation probabilities (e.g., see Chapter 13 of [15]). In any case, this payment is at least the lowest x-coordinate among the x-aligned points to p that have non-zero probability to be allocated to x.

3 The inapproximability result

In this section we present our gap-preserving reduction from MAX-3-LIN(2) to 3OPTIMALAUCTIONDESIGN.

MAX-3-LIN(2): Given a set of n binary variables $v_1, \ldots, v_n$ and a set of m linear equations modulo 2 each containing exactly three variables, find an assignment of the variables that maximizes the number of satisfied equations.

We will first describe the reduction giving the relative location of the points; their exact locations will be defined in Appendix B. Consider an instance I of MAX-3-LIN(2), with n variables v_i, for $i = 1, \ldots, n$ and m linear equations modulo 2, i.e., $e(h) : v_{h_1} + v_{h_2} + v_{h_3} = \alpha_h \pmod 2$, for $h = 1, \ldots, m$, with $1 \leq h_1 < h_2 < h_3 \leq n$ and $\alpha_h \in \{0, 1\}$. Let d_i be the degree of variable v_i, i.e., the number of equations in which v_i participates. Our reduction constructs an instance $R(I)$ of 3OPTIMALAUCTIONDESIGN with a polynomial number of points. The majority of the points in $R(I)$ have coordinates in $(1-\theta, 1+\theta)$ where $\theta \in (0, 1/3600)$ is a very small constant; there are additional points called *blockers* which lie outside (and significantly far from) this region. Both the coordinates of points and their weights are rational numbers that require only polynomial precision. Without loss of generality, we consider weights that do not sum up to 1 and we consistently compute the revenue contributed by (the allocation of) a point as the product of the threshold payment and its weight.

We exploit the following property which has also been used in [1]. Given a linear equation $e(h) : v_i + v_j + v_k = \alpha$, define the four boolean clauses $e_1(h) = (v_i \vee v_j \vee v_k)$, $e_2(h) = (\neg v_i \vee \neg v_j \vee v_k)$, $e_3(h) = (\neg v_i \vee v_j \vee \neg v_k)$, and $e_4(h) = (v_i \vee \neg v_j \vee \neg v_k)$ if $\alpha = 1$ and $e_1(h) = (\neg v_i \vee \neg v_j \vee \neg v_k)$, $e_2(h) = (v_i \vee v_j \vee \neg v_k)$, $e_3(h) = (v_i \vee \neg v_j \vee v_k)$, and $e_4(h) = (\neg v_i \vee v_j \vee v_k)$ if $\alpha = 0$.

Fact 1 *If a linear equation $e(h)$ is true then the four boolean clauses $e_1(h)$, $e_2(h)$, $e_3(h)$, and $e_4(h)$ are true. Otherwise, exactly three of these clauses are true.*

Instance $R(I)$ contains one variable gadget and four clause gadgets per equation (each clause gadget corresponds to one of the clauses mentioned above). The clause gadgets are carefully connected to the variable gadget. We begin by presenting the variable gadget.

The variable gadget. For every variable v_i, $i = 1 \ldots n$, the variable gadget has a *variable point* v_i with weight $d_i(1/2 + 23\theta)$. All these points are x-aligned with $(1, 1, 1)$ and their x-coordinate is in $(1, 1+\theta)$ so that the x-coordinate of v_{i+1} is higher than the x-coordinate of v_i. For each appearance of variable v_i in an equation e, there are four *connection points* $H^i_1(e)$, $H^i_2(e)$, $H^i_3(e)$, and $H^i_4(e)$ with weight θ. Two of these points are y-aligned with v_i; the other two are z-aligned with v_i. The exact location of these points will become clear after the description of the clause gadgets; for the moment, we remark that these points have coordinates in $[1, 1 + \theta)$. We also add a point b with weight θ, at point $(L(b), 1, 1)$ such that $\theta L(b) = 3m(1 + \theta)(1/2 + 23\theta)$.

Point b belongs to a special class of points in our construction which we call *blockers*. The idea is that a blocker can prevent the allocation of a point to a certain player. In particular, b x-blocks the points v_1, ..., v_n in the following sense. It has weight θ (i.e., it corresponds to a highly unlikely valuation vector) but a very large x-coordinate $3m(1+\theta)(1/2+23\theta)/\theta$ to compensate for that (i.e., player x values the item greatly, unlike the other two players). In a revenue-optimal allocation, the only point among v_1, v_2, ..., v_n, and b that is allocated to player x is b. To see why this is true, observe that the contribution of b to the revenue is $3m(1+\theta)(1/2+23\theta)$ whereas by allocating some variable points and b to x, their contribution to the revenue is less than $3m(1+\theta)(1/2+23\theta)$. Furthermore, the weight $d_i(1/2+23\theta)$ of point v_i is significantly high so that in any revenue-optimal allocation, point v_i should be allocated to either player y or player z; these allocations correspond to setting variable v_i to values 1 and 0, respectively. Due to monotonicity, this will enforce the allocation of the y-aligned or z-aligned connection points; intuitively, this will propagate the fact that the variable v_i is set to a certain value to the clause gadgets.

We continue by presenting the clause gadgets and clarify the connection to the variable gadget. For each equation of I, we define four clause gadgets, one for each clause corresponding to the equation. For two x-aligned points p and q, we use the notation $p_{(+x)}q$ and $p_{(-x)}q$ to denote that q has larger and smaller x-coordinate than p, respectively (similarly for the other coordinates). We define the four clause gadgets corresponding to the equation $e(h) : v_i + v_j + v_k = 1 \pmod 2$.

The clause gadget corresponding to clause $e_1(h) = (v_i \vee v_j \vee v_k)$. The clause gadget corresponding to $e_1(h)$ consists of the 5-sequence of points $[A_1(h)_{(+x)}B_1(h)_{(-z)}C_1(h)_{(+x)}D_1(h)_{(-z)}E_1(h)]$. All these points have y-coordinate in $(1-\theta, 1)$, i.e., they lie below the plane $y = 1$. Points $A_1(h)$, $C_1(h)$, and $E_1(h)$ have weight 1; points $B_1(h)$ and $D_1(h)$ have weight θ. The points $A_1(h)$, $C_1(h)$, and $E_1(h)$ are y-aligned to the connection points $H_1^i(h)$, $H_1^j(h)$, and $H_1^k(h)$, respectively. Point $A_1(h)$ is z-blocked by blocker $b_{A_1(h)}$ while point $E_1(h)$ is x-blocked by blocker $b_{E_1(h)}$. See Figure 1.

The z-blocker $b_{A_1(h)}$ has weight θ and a very high z-coordinate equal to $(1+\theta)^2/\theta$. This implies that the highest revenue allocation is the one in which $b_{A_1(h)}$ is allocated to player z and point $A_1(h)$ is not allocated to player z. To see why this is true, observe that the contribution of $b_{A_1(h)}$ to the revenue is $(1+\theta)^2$ whereas by allocating both $A_1(h)$ and $b_{A_1(h)}$ to z or neither of them to z, their contribution to the revenue is less than $(1+\theta)^2$. All blockers in the following have the same coordinate $(1+\theta)^2/\theta$ in the dimension that they block. Consider a monotone allocation in which the connection points $H_1^i(h)$, $H_1^j(h)$, and $H_1^k(h)$ are allocated to players x or z. In this case, we say that the clause gadget is *non-breathing* in the sense that none of the points $A_1(h)$, $C_1(h)$, and $E_1(h)$ can be allocated to player y. Hence, among the monotone allocations in which the clause gadget is non-breathing, the one that maximizes revenue leaves one of $A_1(h)$, $C_1(h)$, and $E_1(h)$ unallocated. In this case, the contribution of these

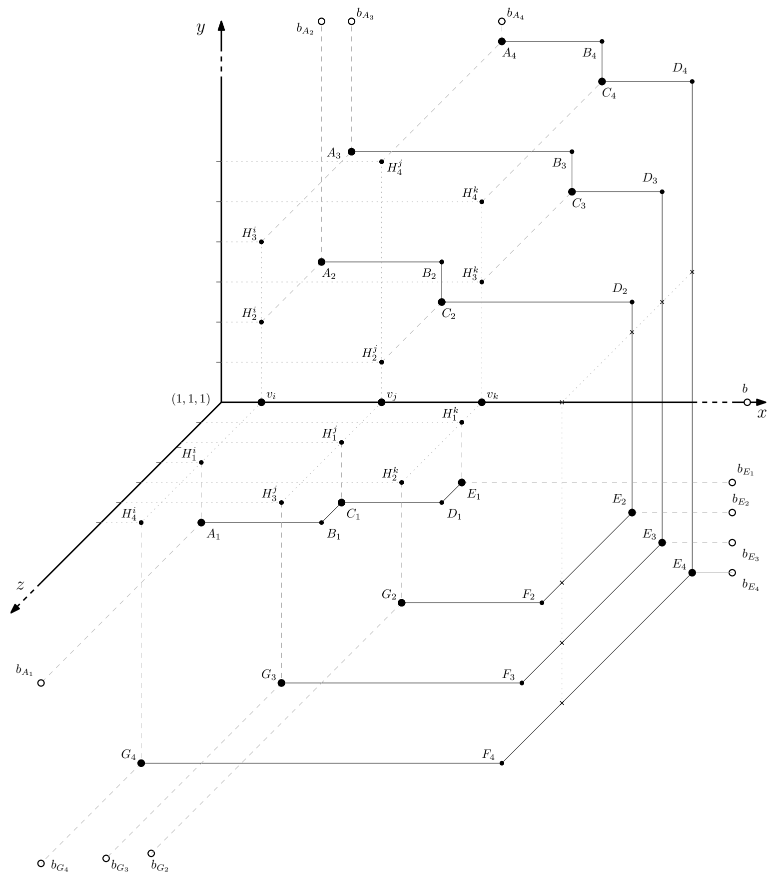

Fig. 1. The clause gadgets corresponding to equation $e(h) : v_i + v_j + v_k = 1 \pmod 2$. Large black disks represent unit-weight or variable points, smaller black disks represent connection points or θ-weight points, whereas white disk represent blockers. Note that the notation of points has been simplified by dropping the index h.

three unit-weight points to revenue is at least 2 and at most $2 + 2\theta$. In contrast, if some of the connection points (say $H_1^j(h)$) is not allocated to players x or z (i.e., the clause gadget is *breathing*), the contribution of $A_1(h)$, $C_1(h)$, and $E_1(h)$ can increase to at least $3 - \theta$ and at most $3 + 2\theta$ by allocating $A_1(h)$ and $B_1(h)$ to player x, $C_1(h)$ and $H_1^j(h)$ to player y, and $D_1(h)$ and $E_1(h)$ to player z. The other cases (i.e., when $H_1^i(h)$ or $H_1^k(h)$ are not allocated to any player) have similar allocations of the same improved revenue.

The clause gadget corresponding to clause $e_2(h) = (\neg v_i \vee \neg v_j \vee v_k)$**.**

The clause gadget corresponding to $e_2(h)$ consists of the 7-sequence of points $[A_2(h)_{(+x)}B_2(h)_{(-y)}C_2(h)_{(+x)}D_2(h)_{(-y)}E_2(h)_{(+z)}F_2(h)_{(-x)}G_2(h)]$. Points $A_2(h)$, $B_2(h)$, $C_2(h)$, $D_2(h)$, and $E_2(h)$ have z-coordinate in $(1-\theta, 1)$, i.e., they lie behind the plane $z = 1$. Points $E_2(h)$, $F_2(h)$, and $G_2(h)$ have y-coordinates in $(1-\theta, 1)$, i.e., they lie below the plane $y = 1$. Points $A_2(h)$, $C_2(h)$, $E_2(h)$ and $G_2(h)$ have weight 1; points $B_2(h)$, $D_2(h)$, and $F_2(h)$ have weight θ. The points $A_2(h)$ and $C_2(h)$ are z-aligned to the connection points $H_2^i(h)$ and $H_2^j(h)$, respectively. Point $G_2(h)$ is y-aligned to connection point $H_2^k(h)$. Point $A_2(h)$ is y-blocked by blocker $b_{A_2(h)}$, point $E_2(h)$ is x-blocked by blocker $b_{E_2(h)}$, and point $G_2(h)$ is z-blocked by blocker $b_{G_2(h)}$. See Figure 1.

Among the monotone allocations in which the connection points $H_2^i(h)$ and $H_2^j(h)$ are allocated to players y or x, and $H_2^k(h)$ is allocated to players x or z (i.e., the clause gadget is non-breathing), the one that maximizes revenue should leave at least one of the points $A_2(h)$, $C_2(h)$, $E_2(h)$, or $G_2(h)$ unallocated. In this case, the total contribution of the unit-weight points to revenue is at least 3 and at most $3+3\theta$. In contrast, if some of the connection points (say $H_2^i(h)$) is not allocated to the players mentioned above, the contribution of $A_2(h)$, $C_2(h)$, $E_2(h)$, and $G_2(h)$ can increase to at least $4-2\theta$ and at most $4+2\theta$ by allocating $A_2(h)$ and $H_2^i(h)$ to player z, $C_2(h)$ and $B_2(h)$ to player y, $E_2(h)$ and $D_2(h)$ to player y, and $G_2(h)$ and $F_2(h)$ to x. The other cases (i.e., when $H_2^j(h)$ or $H_2^k(h)$ are not allocated to any player) have similar allocations of the same improved revenue.

The clause gadgets corresponding to clauses $e_3(h)$ and $e_4(h)$ are analogous and are presented in Appendix B. Observations about potential revenue similar to those for the clause gadget corresponding to clause $e_2(h)$ apply to these cases as well. An important property is that points in different clause gadgets are never aligned. This is achieved by dedicating a distinct xz-plane for the points in the the gadget associated with clause $e_1(h)$ and a distinct xz-plane and a distinct xy-plane for the points in each gadget associated with clause $e_2(h)$, $e_3(h)$, and $e_4(h)$, respectively. The clause gadgets corresponding to clauses of an equation $e(h') : v_i + v_j + v_k = 0 \pmod 2$ are symmetric and have identical properties. The details appear in Appendix B.

We will show that the optimal revenue in $R(I)$ strongly depends on the maximum number of satisfied equations in I. Since approximating the second objective is hard, we will show that approximating the first objective is also hard. We exploit a particular type of monotone allocations.

Definition 1. *We say that an allocation of instance $R(I)$ is* simple *if for every connection point that is allocated to a player, unallocating it violates monotonicity.*

Note that the definition implies that a connection point is allocated to a player only if the allocation of its aligned variable point or clause gadget point also enforces it to be allocated to the same player. We will now explain a relation between simple allocations in $R(I)$ and assignments for I. First observe that, if the variable point v_i is not allocated, then the fact that the allocation is simple implies that all connection points aligned to v_i are not enforced by v_i and, hence, the clause gadgets corresponding to equations in which v_i participates are all breathing. Consider an equation $e(h) : v_i + v_j + v_k = \alpha \pmod 2$, one of its clauses $e_\ell(h)$, the corresponding clause gadget, and an allocation of variable points v_i, v_j, and v_k to players y and z. As mentioned above, we will associate the allocation of a variable point to player y (resp., to player z) with the assignment of value 1 (resp., 0) to its corresponding variable. Then, we can easily verify that the clause gadget associated with $e_\ell(h)$ is breathing if and only if the allocation of the variable points v_i, v_j, and v_k implies an assignment that satisfies clause $e_\ell(h)$. By Fact 1, either the four clause gadgets of $e(h)$ are breathing (if the implied assignment satisfies $e(h)$) or exactly three of them are breathing (if the implied assignment does not satisfy $e(h)$).

Furthermore, when accounting for the revenue of a simple allocation A, we will disregard the revenue obtained by connection points as well as non-blocker points in clause gadgets with weight θ; we will refer to such points as θ-weight points. We refer to the revenue obtained by the remaining points (i.e., variable points, blockers, and unit-weight points in clause gadgets) as *discounted revenue* $\texttt{drev}(A)$. The proofs of the next three lemmas appear in Appendix A.

Lemma 1. *Consider a simple allocation of maximum discounted revenue. If the four clause gadgets corresponding to an equation $e(h)$ are breathing, then their contribution to the discounted revenue is at least $26 + 15\theta + 11\theta^2$ and at most $26 + 30\theta + 11\theta^2$. If only three of them are breathing, the contribution of the four clause gadgets to the discounted revenue is at least $25 + 16\theta + 11\theta^2$ and at most $25 + 31\theta + 11\theta^2$.*

We will call a simple allocation in which all variable points are allocated to players y and z a *complete simple* allocation.

Lemma 2. *The simple allocation with maximum discounted revenue is complete.*

Lemma 3. *For every monotone allocation A with revenue $\texttt{rev}(A)$, there is a complete simple allocation A' such that $\texttt{drev}(A') \geq \texttt{rev}(A) - 46m\theta$.*

Since θ has an extremely small value in our construction, it is clear that the optimal discounted revenue over complete simple allocations is a very good approximation of the optimal revenue over all monotone allocations. The proof of the next lemma exploits this observation.

Lemma 4. *If the maximum number of satisfied equations in I is K, then the revenue in the revenue-optimal monotone allocation of $R(I)$ is between $(28 - 300\theta)m + K$ and $(28 + 300\theta)m + K$.*

Proof. Let us consider an optimal assignment of values to the variables of I so that K linear equations are true. We construct a complete simple allocation A for $R(I)$ as follows. For every variable v_i that is set to 0 (resp., to 1), we allocate the variable point v_i to player z (resp., to player y). In this way, the variable points contribute $\sum_{i=1}^{n} d_i(\frac{1}{2} + 23\theta) \geq 3m/2$ to the discounted revenue of A. The blocker b is allocated to player x and contributes $3m(1+\theta)(\frac{1}{2} + 23\theta) \geq 3m/2$

to the discounted revenue. Then, every clause gadget corresponding to a true (resp., false) clause is breathing (resp., non-breathing). The points in the clause gadgets are allocated so that their contribution to the discounted revenue is as high as possible. By Lemma 1, we have that the contribution of the four breathing clause gadgets associated with an equation to the discounted revenue is at least $26 + 15\theta + 11\theta^2 \geq 26$. For each unsatisfied equation, three of the corresponding clause gadgets are breathing and one is non-breathing. Hence, their contribution to the discounted revenue is at least $25+16\theta+11\theta^2 \geq 25$. So, the total discounted revenue is at least

$$3m/2 + 3m/2 + 26K + 25(m - K) \geq (28 - 300\theta)m + K.$$

Clearly, the right-hand side of this inequality is a lower bound on the revenue of the revenue-optimal monotone allocation as well.

Now, consider a complete simple allocation of maximum discounted revenue and the assignment of values to the variables v_i this allocation implies. Consider the equations in which the four corresponding clause gadgets are all breathing. By our construction, this implies that the corresponding equations are satisfied by the assignment; so, there are at most K such quadruples and the remaining $m - K$ quadruples of clause gadgets will have three breathing and one non-breathing clause gadgets. In total, using Lemma 1, the discounted revenue of the complete simple allocation is at most

$$3m(\frac{1}{2}+23\theta)+3m(1+\theta)(\frac{1}{2}+23\theta)+K(26+30\theta+11\theta^2)+(m-K)(25+31\theta+11\theta^2)$$
$$< (28 + 251\theta)m + K.$$

Hence, by Lemma 3, the revenue-optimal monotone allocation has revenue at most $(28 + 300\theta)m + K$. $\square$

We are ready to prove our main result.

Theorem 1. *For every constant $\delta \in (0, 1/2)$, it is NP-hard to approximate* 3OptimalAuctionDesign *within a factor $\frac{57+\delta}{58-\delta}$.*

Proof. Let $\delta \in (0, 1/2)$, $\eta = \delta/3$ and $\theta = \delta/1800$. Since, given I, it is NP-hard to distinguish cases where the maximum number of satisfied equations is at least $(1 - \eta)m$ or at most $(1/2 + \eta)m$ [9], Lemma 4 implies that it is NP-hard to distinguish between cases where the maximum revenue among all monotone allocations of $R(I)$ is at least $(28 - 300\theta)m + (1 - \eta)m = (58 - \delta)m/2$ and at most $(28 + 300\theta)m + (1/2 + \eta)m = (57 + \delta)m/2$ is NP-hard as well. $\square$

4 Deterministic vs randomized auctions

We now present the upper bound on the revenue-gap between deterministic and randomized mechanism.

Theorem 2. *The revenue obtained by the optimal deterministic truthful mechanism can be at most* $\frac{13}{14}$ *of the revenue obtained by the optimal truthful-in-expectation mechanism.*

Proof. Our construction consists of 22 points described in Table 1 (see also Figure 3 in Appendix C). The parameter θ is positive and arbitrarily small. Point

Table 1. The construction in the proof of Theorem 2.

Point	c_x	c_y	c_z	wgt	Point	c_x	c_y	c_z	wgt
q_1	1	$1+\theta$	1	θ	p_1	$1+\theta$	1	1	θ
q_2	1	$1+\theta$	$1-\theta$	1	p_2	$1+\theta$	$1-\theta$	1	1
$b(q_2)$	1	$(1+\theta)^2/\theta$	$1-\theta$	θ	$b(p_2)$	$(1+\theta)^2/\theta$	$1-\theta$	1	θ
q_3	$1+\theta/3$	$1+\theta$	$1-\theta$	θ	p_3	$1+\theta$	$1-\theta$	$1+\theta/2$	θ
q_4	$1+\theta/3$	$1+\theta/2$	$1-\theta$	1	p_4	$1+2\theta/3$	$1-\theta$	$1+\theta/2$	1
$b(q_4)$	$(1+\theta)^2/\theta$	$1+\theta/2$	$1-\theta$	θ	$b(p_4)$	$1+2\theta/3$	$1-\theta$	$(1+\theta)^2/\theta$	θ
q_5	$1+\theta/3$	$1+\theta/2$	$1-\theta/2$	θ	p_5	$1+2\theta/3$	$1-\theta/2$	$1+\theta/2$	θ
q_6	1	$1+\theta/2$	$1-\theta/2$	1	p_6	$1+2\theta/3$	$1-\theta/2$	1	1
$b(q_6)$	1	$(1+\theta)^2/\theta$	$1-\theta/2$	θ	$b(p_6)$	$(1+\theta)^2/\theta$	$1-\theta/2$	1	θ
q_7	1	$1+\theta/2$	1	θ	p_7	$1+2\theta/3$	1	1	θ
u	1	1	1	1	$b(u)$	1	1	$(1+\theta)^2/\theta$	θ

$b(u)$ is a z-blocker for point u. The existence of $b(u)$ guarantees that in a revenue maximizing allocation $b(u)$ will be allocated to player z and point u should not be allocated to z. A similar observation applies for blockers $b(q_2)$, $b(q_4)$, $b(q_6)$, $b(p_2)$, $b(p_4)$, and $b(p_6)$ and their corresponding blocked points. Regarding the remaining points, we refer to the ones of weight 1 as heavy points, and to the ones of weight θ as light points.

Consider the randomized allocation in which the blockers are allocated to their preferred direction, the heavy points are allocated equiprobably between their two non-blocked directions, and the light points are allocated as follows: points q_1, q_7, and p_5 are allocated equiprobably between players y and z, points p_1, p_7, and q_3 are allocated equiprobably between players x and y, and points q_5 and p_3 are allocated equiprobably between players x and z. It can be easily verified that this is a monotone allocation with revenue at least $7(1+\theta)^2+(7+8\theta)(1-\theta)\geq 14$.

Now, let us examine the possible deterministic allocations. If u is not allocated the maximum revenue does not exceed $7(1+\theta)^2+(6+8\theta)(1+\theta)\leq 13+28\theta+15\theta^2$. Otherwise, if u is allocated to player y or z, some of the other heavy points can not be allocated to a non-blocked direction/player. To see why this is true, assume that u is allocated to player y (the case that u is allocated to player z is symmetric). Then, points q_2 and q_6 can only be allocated to player x in a monotone allocation, thus point q_4 can not be allocated to any of its non-blocked directions. Again, the maximum revenue does not exceed $13+28\theta+15\theta^2$. The theorem follows since θ can take any arbitrarily small positive value. □

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A Omitted proofs

Proof of Lemma 1. The proof follows by accounting for the revenue from the breathing and non-breathing clause gadgets (see the description in Section 3 and in Appendix B). For example, the unit-weight points in four breathing clause gadgets $e_1(h)$, $e_2(h)$, $e_3(h)$, and $e_4(h)$ corresponding to equation $e(h) : v_i + v_j + v_k = \alpha \pmod 2$ contribute at most $3+2\theta$, $4+2\theta$, $4+2\theta$, and $4+2\theta$, respectively, and at least $3-\theta$, $4-2\theta$, $4-2\theta$, and $4-2\theta$, respectively. Each of the eleven blockers in these gadgets contribute exactly $(1+\theta)^2$ to the revenue. This yields the first two bounds for $\alpha = 1$. The remaining two bounds follow similarly. □

Proof of Lemma 2. Consider a non-complete simple allocation in which variable point v_i has not been allocated to any player. We will allocate the variable point v_i to the player among y and z that maximizes the number of breathing clause gadgets. Clearly, for every equation $e(h) : v_i + v_j + v_k = \alpha \pmod 2$ such that some of the variable points v_j and v_k are not allocated, the four clause gadgets will remain breathing no matter how the variable point v_i is allocated. For every equation $e(h) : v_i + v_j + v_k = \alpha \pmod 2$ such that variable points v_j and v_k are allocated (to players y or z), there is an allocation of v_i to either player y or player z that leaves the four clause gadgets corresponding to $e(h)$ breathing. Hence, there is an allocation of v_i such that at most $d_i/2$ quadruples of clause gadgets have a non-breathing gadget. By Lemma 1, the contribution from these quadruples of gadgets to the discounted revenue decreases by at most $(1 + 14\theta)d_i/2$. The contribution of the remaining (at most d_i) quadruples of gadgets decreases by at most $15\theta d_i$. Clearly, the total decrease in discounted revenue is smaller than the additional revenue $d_i(1/2 + 23\theta)$ obtained by the allocation of v_i. □

Proof of Lemma 3. Given the monotone allocation A, we construct a simple allocation A'' by allocating variable points, clause gadget points, and blockers as in A. Among the connection points, the only ones that are allocated to some player (like in A) are the ones whose allocation is enforced. By Lemma 2, there is a complete simple allocation A' with discounted revenue $\texttt{drev}(A') \geq \texttt{drev}(A'')$. The lemma follows since the total number of connection points and θ-weight points in clause gadgets is $23m$ and the contribution of each of them to the revenue of A cannot exceed $\theta(1 + \theta) < 2\theta$. □

B The omitted gadgets in the construction

We proceed by presenting the remaining two clause gadgets for the equation $e(h) : v_i + v_j + v_k = 1 \pmod 2$.

The clause-gadget corresponding to clause $e_3(h) = (\neg v_i \vee v_j \vee \neg v_k)$.

The clause-gadget corresponding to $e_3(h)$ consists of the 7-sequence of points $[A_3(h)_{(+x)}B_3(h)_{(-y)}C_3(h)_{(+x)}D_3(h)_{(-y)}E_3(h)_{(+z)}F_3(h)_{(-x)}G_3(h)]$. Points $A_3(h)$, $B_3(h)$, $C_3(h)$, $D_3(h)$, and $E_3(h)$ have z-coordinate in $(1-\theta, 1)$, i.e., they lie behind the plane $z = 1$. Points $E_3(h)$, $F_3(h)$, and $G_3(h)$ have y-coordinates in $(1-\theta, 1)$, i.e., they lie below the plane $y = 1$. Points $A_3(h)$, $C_3(h)$, $E_3(h)$ and $G_3(h)$ have weight 1; points $B_3(h)$, $D_3(h)$, and $F_3(h)$ have weight θ. The points $A_3(h)$ and $C_3(h)$ are z-aligned to the points $H_3^i(h)$ and $H_3^k(h)$, respectively. Point $G_3(h)$ is y-aligned to point $H_3^j(h)$. Point $A_3(h)$ is y-blocked by blocker $b_{A_3(h)}$, point $E_3(h)$ is x-blocked by blocker $b_{E_3(h)}$, and point $G_3(h)$ is z-blocked by blocker $b_{G_3(h)}$. See Figure 1.

Among the monotone allocations in which the connection points $H_3^i(h)$ and $H_3^k(h)$ are allocated to players y or x, and $H_3^j(h)$ is allocated to players x or z (i.e., the clause gadget is non-breathing), the one that maximizes revenue should leave at least one of the points $A_3(h)$, $C_3(h)$, $E_3(h)$, or $G_3(h)$ unallocated. In this case, the total contribution of the unit-weight points to revenue is at least 3 and at most $3+3\theta$. In contrast, if some of the connection points (say $H_3^i(h)$) is not allocated to the players mentioned above, the contribution of $A_3(h)$, $C_3(h)$, $E_3(h)$, and $G_3(h)$ can increase to at least $4-2\theta$ and at most $4+2\theta$ by allocating $A_3(h)$ and $H_3^i(h)$ to player z, $C_3(h)$ and $B_3(h)$ to player y, $E_3(h)$ and $D_3(h)$ to player y, and $G_3(h)$ and $F_3(h)$ to x. The other cases (i.e., when $H_3^j(h)$ or $H_3^k(h)$ are not allocated to any player) have similar allocations of the same improved revenue.

The clause-gadget corresponding to clause $e_4(h) = (v_i \vee \neg v_j \vee \neg v_k)$.
The clause-gadget corresponding to $e_4(h)$ consists of the 7-sequence of points $[A_4(h)_{(+x)}B_4(h)_{(-y)}C_4(h)_{(+x)}D_4(h)_{(-y)}E_4(h)_{(+z)}F_4(h)_{(-x)}G_4(h)]$. Points $A_4(h)$, $B_4(h)$, $C_4(h)$, $D_4(h)$, and $E_4(h)$ have z-coordinate in $(1-\theta, 1)$, i.e., they lie behind the plane $z = 1$. Points $E_4(h)$, $F_4(h)$, and $G_4(h)$ have y-coordinates in $(1-\theta, 1)$, i.e., they lie below the plane $y = 1$. Points $A_4(h)$, $C_4(h)$, $E_4(h)$ and $G_4(h)$ have weight 1; points $B_4(h)$, $D_4(h)$, and $F_4(h)$ have weight θ. The points $A_4(h)$ and $C_4(h)$ are z-aligned to the points $H_4^j(h)$ and $H_4^k(h)$, respectively. Point $G_4(h)$ is y-aligned to point $H_4^i(h)$. Point $A_4(h)$ is y-blocked by blocker $b_{A_4(h)}$, point $E_4(h)$ is x-blocked by blocker $b_{E_4(h)}$, and point $G_4(h)$ is z-blocked by blocker $b_{G_4(h)}$. See Figure 1.

Among the monotone allocations in which the connection points $H_4^j(h)$ and $H_4^k(h)$ are allocated to players y or x, and $H_4^i(h)$ is allocated to players x or z (i.e., the clause gadget is non-breathing), the one that maximizes revenue should leave at least one of the points $A_4(h)$, $C_4(h)$, $E_4(h)$, or $G_4(h)$ unallocated. In this case, the total contribution of the unit-weight points to revenue is at least 3 and at most $3+3\theta$. In contrast, if some of the connection points (say $H_4^i(h)$) is not allocated to the players mentioned above, the contribution of $A_4(h)$, $C_4(h)$, $E_4(h)$, and $G_4(h)$ can increase to at least $4-2\theta$ and at most $4+2\theta$ by allocating $G_4(h)$ and $H_4^i(h)$ to player y, $A_4(h)$ and $B_4(h)$ to player x, $C_4(h)$ and $D_4(h)$ to player x, and $E_4(h)$ and $F_4(h)$ to z. The other cases (i.e., when $H_4^j(h)$ or $H_4^k(h)$ are not allocated to any player) have similar allocations of the same improved revenue.

The four clause gadgets for the equation $e(h) : v_i + v_j + v_k = 0 \pmod 2$ are as follows.

The clause-gadget corresponding to clause $e_1(h) = (\neg v_i \vee \neg v_j \vee \neg v_k)$**.** The clause-gadget corresponding to $e_1(h)$ consists of the 5-sequence of points $[A_1(h)_{(+x)}B_1(h)_{(-y)}C_1(h)_{(+x)}D_1(h)_{(-y)}E_1(h)]$. All these points have z-coordinate in $(1-\theta, 1)$, i.e., they lie behind the plane $z = 1$. Points $A_1(h)$, $C_1(h)$, and $E_1(h)$ have weight 1; points $B_1(h)$ and $D_1(h)$ have weight θ. The points $A_1(h)$, $C_1(h)$, and $E_1(h)$ are z-aligned to the points $H_1^i(h)$, $H_1^j(h)$, and $H_1^k(h)$, respectively. Point $A_1(h)$ is y-blocked by blocker $b_{A_1(h)}$ while point $E_1(h)$ is x-blocked by blocker $b_{E_1(h)}$. See Figure 2.

Among the allocations in which the connection points $H_1^i(h)$, $H_1^j(h)$ and $H_1^k(h)$ are allocated to players y or x (i.e., the clause gadget is non-breathing), the one that maximizes revenue leaves one of $A_1(h)$, $C_1(h)$, and $E_1(h)$ unallocated. In this case, their contribution to revenue is at least 2 and at most $2 + 2\theta$. In contrast, if some of the connection points (say $H_1^j(h)$) is not allocated to players x or y (i.e., the clause gadget is breathing), the contribution of $A_1(h)$, $C_1(h)$, and $E_1(h)$ can increase to at least $3 - \theta$ and at most $3 + 2\theta$ by allocating $A_1(h)$ and $B_1(h)$ to player x, $C_1(h)$ and $H_1^j(h)$ to z, and $D_1(h)$ and $E_1(h)$ to y. The other cases (i.e., when $H_1^i(h)$ or $H_1^k(h)$ are not allocated to any player) have similar allocations of the same improved revenue.

The clause-gadget corresponding to clause $e_2(h) = (v_i \vee v_j \vee \neg v_k)$**.** The clause-gadget corresponding to $e_2(h)$ consists of the 7-sequence of points $[A_2(h)_{(+x)}B_2(h)_{(-z)}C_2(h)_{(+x)}D_2(h)_{(-z)}E_2(h)_{(+y)}F_2(h)_{(-x)}G_2(h)]$. Points $A_2(h)$, $B_2(h)$, $C_2(h)$, $D_2(h)$, and $E_2(h)$ have y-coordinate in $(1 - \theta, 1)$, i.e., they lie below the plane $y = 1$. Points $E_2(h)$, $F_2(h)$, and $G_2(h)$ have z-coordinates in $(1 - \theta, 1)$, i.e., they lie behind the plane $z = 1$. Points $A_2(h)$, $C_2(h)$, $E_2(h)$ and $G_2(h)$ have weight 1; points $B_2(h)$, $D_2(h)$, and $F_2(h)$ have weight θ. The points $A_2(h)$ and $C_2(h)$ are y-aligned to the points $H_2^i(h)$ and $H_2^j(h)$, respectively. Point $G_2(h)$ is z-aligned to point $H_2^k(h)$. Point $A_2(h)$ is z-blocked by blocker $b_{A_2(h)}$, point $E_2(h)$ is x-blocked by blocker $b_{E_2(h)}$, and point $G_2(h)$ is y-blocked by blocker $b_{G_2(h)}$. See Figure 2.

Among the monotone allocations in which the connection points $H_2^i(h)$ and $H_2^j(h)$ are allocated to players x or z, and $H_2^k(h)$ is allocated to players x or y (i.e., the clause gadget is non-breathing), the one that maximizes revenue should leave at least one of the points $A_2(h)$, $C_2(h)$, $E_2(h)$, or $G_2(h)$ unallocated. In this case, the total contribution of the unit-weight points to revenue is at least 3 and at most $3 + 3\theta$. In contrast, if some of the connection points (say $H_2^i(h)$) is not allocated to the players mentioned above, the contribution of $A_2(h)$, $C_2(h)$, $E_2(h)$, and $G_2(h)$ can increase to at least $4 - 2\theta$ and at most $4 + 2\theta$ by allocating $A_2(h)$ and $H_2^i(h)$ to player y, $C_2(h)$ and $B_2(h)$ to player z, $E_2(h)$ and $D_2(h)$ to player z, and $G_2(h)$ and $F_2(h)$ to x. The other cases (i.e., when $H_2^j(h)$ or $H_2^k(h)$ are not allocated to any player) have similar allocations of the same improved revenue.

The clause-gadget corresponding to clause $e_3(h) = (v_i \vee \neg v_j \vee v_k)$**.** The clause-gadget corresponding to $e_3(h)$ consists of the 7-sequence of points

$[A_3(h)_{(+x)}B_3(h)_{(-z)}C_3(h)_{(+x)}D_3(h)_{(-z)}E_3(h)_{(+y)}F_3(h)_{(-x)}G_3(h)]$. Points $A_3(h)$, $B_3(h)$, $C_3(h)$, $D_3(h)$, and $E_3(h)$ have y-coordinate in $(1-\theta, 1)$, i.e., they lie below the plane $y = 1$. Points $E_3(h)$, $F_3(h)$, and $G_3(h)$ have z-coordinates in $(1-\theta, 1)$, i.e., they lie behind the plane $z = 1$. Points $A_3(h)$, $C_3(h)$, $E_3(h)$ and $G_3(h)$ have weight 1; points $B_3(h)$, $D_3(h)$, and $F_3(h)$ have weight θ. The points $A_3(h)$ and $C_3(h)$ are y-aligned to the points $H_3^i(h)$ and $H_3^k(h)$, respectively. Point $G_3(h)$ is z-aligned to point $H_3^j(h)$. Point $A_3(h)$ is z-blocked by blocker $b_{A_3(h)}$, point $E_3(h)$ is x-blocked by blocker $b_{E_3(h)}$, and point $G_3(h)$ is y-blocked by blocker $b_{G_3(h)}$. See Figure 2.

Among the monotone allocations in which the connection points $H_3^i(h)$ and $H_3^k(h)$ are allocated to players x or z, and $H_3^j(h)$ is allocated to players x or y (i.e., the clause gadget is non-breathing), the one that maximizes revenue should leave at least one of the points $A_3(h)$, $C_3(h)$, $E_3(h)$, or $G_3(h)$ unallocated. In this case, the total contribution of the unit-weight points to revenue is at least 3 and at most $3+3\theta$. In contrast, if some of the connection points (say $H_3^k(h)$) is not allocated to the players mentioned above, the contribution of $A_3(h)$, $C_3(h)$, $E_3(h)$, and $G_3(h)$ can increase to at least $4-2\theta$ and at most $4+2\theta$ by allocating $A_3(h)$ and $B_3(h)$ to player x, $C_3(h)$ and $H_3^i(h)$ to player y, $E_3(h)$ and $D_3(h)$ to player z, and $G_3(h)$ and $F_3(h)$ to x. The other cases (i.e., when $H_3^i(h)$ or $H_3^j(h)$ are not allocated to any player) have similar allocations of the same improved revenue.

The clause-gadget corresponding to clause $e_4(h) = (\neg v_i \vee v_j \vee v_k)$. The clause-gadget corresponding to $e_4(h)$ consists of the 7-sequence of points $[A_4(h)_{(+x)}B_4(h)_{(-z)}C_4(h)_{(+x)}D_4(h)_{(-z)}E_4(h)_{(+y)}F_4(h)_{(-x)}G_4(h)]$. Points $A_4(h)$, $B_4(h)$, $C_4(h)$, $D_4(h)$, and $E_4(h)$ have y-coordinate in $(1-\theta, 1)$, i.e., they lie below the plane $y = 1$. Points $E_4(h)$, $F_4(h)$, and $G_4(h)$ have z-coordinates in $(1-\theta, 1)$, i.e., they lie behind the plane $z = 1$. Points $A_4(h)$, $C_4(h)$, $E_4(h)$ and $G_4(h)$ have weight 1; points $B_4(h)$, $D_4(h)$, and $F_4(h)$ have weight θ. The points $A_4(h)$ and $C_4(h)$ are y-aligned to the points $H_4^j(h)$ and $H_4^k(h)$, respectively. Point $G_4(h)$ is z-aligned to point $H_4^i(h)$. Point $A_4(h)$ is z-blocked by blocker $b_{A_4(h)}$, point $E_4(h)$ is x-blocked by blocker $b_{E_4(h)}$, and point $G_4(h)$ is y-blocked by blocker $b_{G_4(h)}$. See Figure 2.

Among the monotone allocations in which the connection points $H_4^j(h)$ and $H_4^k(h)$ are allocated to players x or z, and $H_4^i(h)$ is allocated to players x or y (i.e., the clause gadget is non-breathing), the one that maximizes revenue should leave at least one of the points $A_4(h)$, $C_4(h)$, $E_4(h)$, or $G_4(h)$ unallocated. In this case, the total contribution of the unit-weight points to revenue is at least 3 and at most $3+3\theta$. In contrast, if some of the connection points (say $H_4^i(h)$) is not allocated to the players mentioned above, the contribution of $A_4(h)$, $C_4(h)$, $E_4(h)$, and $G_4(h)$ can increase to at least $4-2\theta$ and at most $4+2\theta$ by allocating $A_4(h)$ and $B_4(h)$ to player x, $C_4(h)$ and $D_4(h)$ to player x, $E_4(h)$ and $F_4(h)$ to player y, and $G_4(h)$ and $H_4^i(h)$ to z. The other cases (i.e., when $H_4^j(h)$ or $H_4^k(h)$ are not allocated to any player) have similar allocations of the same improved revenue.

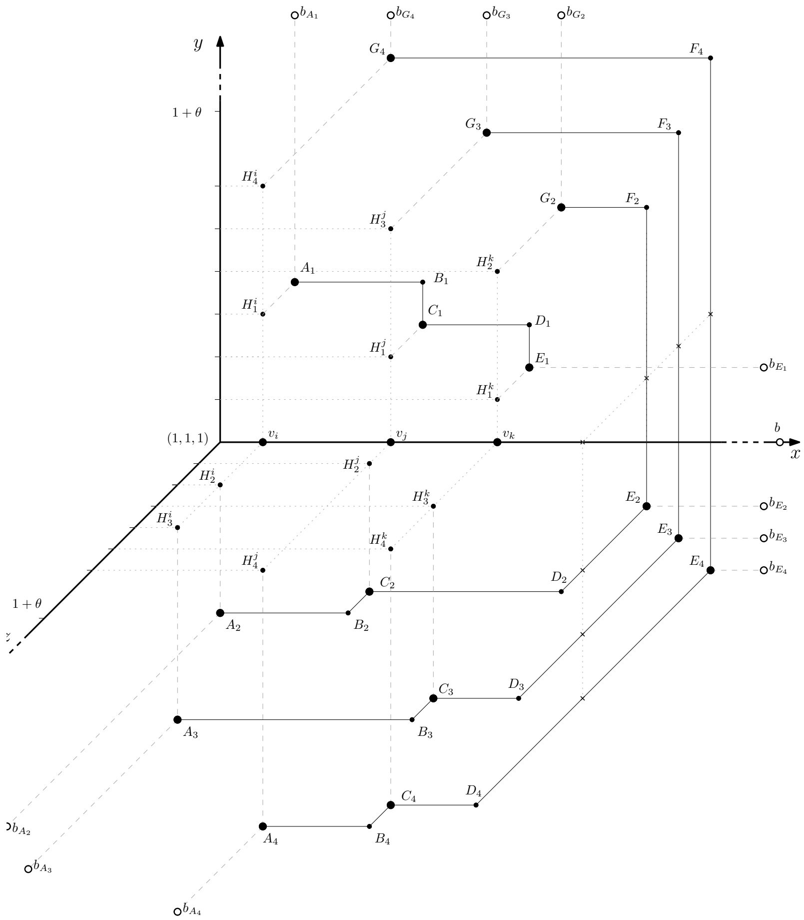

Fig. 2. The clause gadgets corresponding to equation $e(h) : v_i + v_j + v_k = 0 \pmod 2$. Large black disks represent unit-weight or variable points, smaller black disks represent connection points or θ-weight points, whereas white disk represent blockers. Note that the notation of points has been simplified by dropping the index h.

In order to complete the formal description of our construction, we give the exact locations of all points. Let $\epsilon = \theta / \max\{n+1, m+1\}$. The variable point v_i is located at $(1+i\epsilon, 1, 1)$ and (as discussed in Section 3) has weight $d_i(1/2+23\theta)$. The blocker b is located at $(3m(1+\theta)(1/2+23\theta)/\theta, 1, 1)$ and has weight θ. The exact location of the remaining points (points in clause gadgets and connection points) are presented in the following eight tables.

Table 2. The points in the clause gadget associated with the clause $e_1(h) = (v_i \vee v_j \vee v_k)$ of the equation $e(h) : v_i + v_j + v_k = 1 \pmod 2$.

Point	c_x	c_y	c_z	Weight
$A_1(h)$	$1+i\epsilon$	$1-h\epsilon$	$1+h\epsilon+2/6\epsilon$	1
$B_1(h)$	$1+j\epsilon$	$1-h\epsilon$	$1+h\epsilon+2/6\epsilon$	θ
$C_1(h)$	$1+j\epsilon$	$1-h\epsilon$	$1+h\epsilon+1/6\epsilon$	1
$D_1(h)$	$1+k\epsilon$	$1-h\epsilon$	$1+h\epsilon+1/6\epsilon$	θ
$E_1(h)$	$1+k\epsilon$	$1-h\epsilon$	$1+h\epsilon$	1
$b_{A_1(h)}$	$1+i\epsilon$	$1-h\epsilon$	$(1+\theta)^2/\theta$	θ
$b_{E_1(h)}$	$(1+\theta)^2/\theta$	$1-h\epsilon$	$1+h\epsilon$	θ
$H_1^i(h)$	$1+i\epsilon$	1	$1+h\epsilon+2\epsilon/6$	θ
$H_1^j(h)$	$1+j\epsilon$	1	$1+h\epsilon+1\epsilon/6$	θ
$H_1^k(h)$	$1+k\epsilon$	1	$1+h\epsilon$	θ

Table 3. The points in the clause gadget associated with the clause $e_2(h) = (\neg v_i \vee \neg v_j \vee v_k)$ of the equation $e(h) : v_i + v_j + v_k = 1 \pmod 2$.

Point	c_x	c_y	c_z	Weight
$A_2(h)$	$1+i\epsilon$	$1+h\epsilon+\epsilon/6$	$1-h\epsilon-\epsilon/4$	1
$B_2(h)$	$1+j\epsilon$	$1+h\epsilon+\epsilon/6$	$1-h\epsilon-\epsilon/4$	θ
$C_2(h)$	$1+j\epsilon$	$1+h\epsilon$	$1-h\epsilon-\epsilon/4$	1
$D_2(h)$	$1+\theta$	$1+h\epsilon$	$1-h\epsilon-\epsilon/4$	θ
$E_2(h)$	$1+\theta$	$1-h\epsilon-\epsilon/4$	$1-h\epsilon-\epsilon/4$	1
$F_2(h)$	$1+\theta$	$1-h\epsilon-\epsilon/4$	$1+h\epsilon+3\epsilon/6$	θ
$G_2(h)$	$1+k\epsilon$	$1-h\epsilon-\epsilon/4$	$1+h\epsilon+3\epsilon/6$	1
$b_{A_2(h)}$	$1+i\epsilon$	$(1+\theta)^2/\theta$	$1-h\epsilon-\epsilon/4$	θ
$b_{E_2(h)}$	$(1+\theta)^2/\theta$	$1-h\epsilon-\epsilon/4$	$1-h\epsilon-\epsilon/4$	θ
$b_{G_2(h)}$	$1+k\epsilon$	$1-h\epsilon-\epsilon/4$	$(1+\theta)^2/\theta$	θ
$H_2^i(h)$	$1+i\epsilon$	$1+h\epsilon+\epsilon/6$	1	θ
$H_2^j(h)$	$1+j\epsilon$	$1+h\epsilon$	1	θ
$H_2^k(h)$	$1+k\epsilon$	1	$1+h\epsilon+3\epsilon/6$	θ

Table 4. The points in the clause gadget associated with the clause $e_3(h) = (\neg v_i \vee v_j \vee \neg v_k)$ of the equation $e(h) : v_i + v_j + v_k = 1 \pmod 2$.

Point	c_x	c_y	c_z	Weight
$A_3(h)$	$1 + i\epsilon$	$1 + h\epsilon + 3\epsilon/6$	$1 - h\epsilon - 2\epsilon/4$	1
$B_3(h)$	$1 + k\epsilon$	$1 + h\epsilon + 3\epsilon/6$	$1 - h\epsilon - 2\epsilon/4$	θ
$C_3(h)$	$1 + k\epsilon$	$1 + h\epsilon + 2\epsilon/6$	$1 - h\epsilon - 2\epsilon/4$	1
$D_3(h)$	$1 + \theta$	$1 + h\epsilon + 2\epsilon/6$	$1 - h\epsilon - 2\epsilon/4$	θ
$E_3(h)$	$1 + \theta$	$1 - h\epsilon - 2\epsilon/4$	$1 - h\epsilon - 2\epsilon/4$	1
$F_3(h)$	$1 + \theta$	$1 - h\epsilon - 2\epsilon/4$	$1 + h\epsilon + 4\epsilon/6$	θ
$G_3(h)$	$1 + j\epsilon$	$1 - h\epsilon - 2\epsilon/4$	$1 + h\epsilon + 4\epsilon/6$	1
$b_{A_3(h)}$	$1 + i\epsilon$	$(1+\theta)^2/\theta$	$1 - h\epsilon - 2\epsilon/4$	θ
$b_{E_3(h)}$	$(1+\theta)^2/\theta$	$1 - h\epsilon - 2\epsilon/4$	$1 - h\epsilon - 2\epsilon/4$	θ
$b_{G_3(h)}$	$1 + j\epsilon$	$1 - h\epsilon - 2\epsilon/4$	$(1+\theta)^2/\theta$	θ
$H_3^i(h)$	$1 + i\epsilon$	$1 + h\epsilon + 3\epsilon/6$	1	θ
$H_3^j(h)$	$1 + j\epsilon$	1	$1 + h\epsilon + 4\epsilon/6$	θ
$H_3^k(h)$	$1 + k\epsilon$	$1 + h\epsilon + 2\epsilon/6$	1	θ

Table 5. The points in the clause gadget associated with the clause $e_4(h) = (v_i \vee \neg v_j \vee \neg v_k)$ of the equation $e(h) : v_i + v_j + v_k = 1 \pmod 2$.

Point	c_x	c_y	c_z	Weight
$A_4(h)$	$1 + j\epsilon$	$1 + h\epsilon + 5\epsilon/6$	$1 - h\epsilon - 3\epsilon/4$	1
$B_4(h)$	$1 + k\epsilon$	$1 + h\epsilon + 5\epsilon/6$	$1 - h\epsilon - 3\epsilon/4$	θ
$C_4(h)$	$1 + k\epsilon$	$1 + h\epsilon + 4\epsilon/6$	$1 - h\epsilon - 3\epsilon/4$	1
$D_4(h)$	$1 + \theta$	$1 + h\epsilon + 4\epsilon/6$	$1 - h\epsilon - 3\epsilon/4$	θ
$E_4(h)$	$1 + \theta$	$1 - h\epsilon - 3\epsilon/4$	$1 - h\epsilon - 3\epsilon/4$	1
$F_4(h)$	$1 + \theta$	$1 - h\epsilon - 3\epsilon/4$	$1 + h\epsilon + 5\epsilon/6$	θ
$G_4(h)$	$1 + i\epsilon$	$1 - h\epsilon - 3\epsilon/4$	$1 + h\epsilon + 5\epsilon/6$	1
$b_{A_4(h)}$	$1 + j\epsilon$	$(1+\theta)^2/\theta$	$1 - h\epsilon - 3\epsilon/4$	θ
$b_{E_4(h)}$	$(1+\theta)^2/\theta$	$1 - h\epsilon - 3\epsilon/4$	$1 - h\epsilon - 3\epsilon/4$	θ
$b_{G_4(h)}$	$1 + i\epsilon$	$1 - h\epsilon - 3\epsilon/4$	$(1+\theta)^2/\theta$	θ
$H_4^i(\mathrm{h})$	$1 + i\epsilon$	1	$1 + h\epsilon + 5\epsilon/6$	θ
$H_4^j(h)$	$1 + j\epsilon$	$1 + h\epsilon + 5\epsilon/6$	1	θ
$H_4^k(h)$	$1 + k\epsilon$	$1 + h\epsilon + 4\epsilon/6$	1	θ

Table 6. The points in the clause gadget associated with the clause $e_1(h) = (\neg v_i \vee \neg v_j \vee \neg v_k)$ of the equation $e(h) : v_i + v_j + v_k = 0 \pmod 2$.

Point	c_x	c_y	c_z	Weight
$A_1(h)$	$1 + i\epsilon$	$1 + h\epsilon + 2/6\epsilon$	$1 - h\epsilon$	1
$B_1(h)$	$1 + j\epsilon$	$1 + h\epsilon + 2/6\epsilon$	$1 - h\epsilon$	θ
$C_1(h)$	$1 + j\epsilon$	$1 + h\epsilon + 1/6\epsilon$	$1 - h\epsilon$	1
$D_1(h)$	$1 + k\epsilon$	$1 + h\epsilon + 1/6\epsilon$	$1 - h\epsilon$	θ
$E_1(h)$	$1 + k\epsilon$	$1 + h\epsilon$	$1 - h\epsilon$	1
$b_{A_1(h)}$	$1 + i\epsilon$	$(1+\theta)^2/\theta$	$1 - h\epsilon$	θ
$b_{E_1(h)}$	$(1+\theta)^2/\theta$	$1 + h\epsilon$	$1 - h\epsilon$	θ
$H_1^i(h)$	$1 + i\epsilon$	$1 + h\epsilon + 2\epsilon/6$	1	θ
$H_1^j(h)$	$1 + j\epsilon$	$1 + h\epsilon + 1\epsilon/6$	1	θ
$H_1^k(h)$	$1 + k\epsilon$	$1 + h\epsilon$	1	θ

Table 7. The points in the clause gadget associated with the clause $e_2(h) = (v_i \vee v_j \vee \neg v_k)$ of the equation $e(h) : v_i + v_j + v_k = 0 \pmod 2$.

Point	c_x	c_y	c_z	Weight
$A_2(h)$	$1 + i\epsilon$	$1 - h\epsilon - \epsilon/4$	$1 + h\epsilon + \epsilon/6$	1
$B_2(h)$	$1 + j\epsilon$	$1 - h\epsilon - \epsilon/4$	$1 + h\epsilon + \epsilon/6$	θ
$C_2(h)$	$1 + j\epsilon$	$1 - h\epsilon - \epsilon/4$	$1 + h\epsilon$	1
$D_2(h)$	$1 + \theta$	$1 - h\epsilon - \epsilon/4$	$1 + h\epsilon$	θ
$E_2(h)$	$1 + \theta$	$1 - h\epsilon - \epsilon/4$	$1 - h\epsilon - \epsilon/4$	1
$F_2(h)$	$1 + \theta$	$1 + h\epsilon + 3\epsilon/6$	$1 - h\epsilon - \epsilon/4$	θ
$G_2(h)$	$1 + k\epsilon$	$1 + h\epsilon + 3\epsilon/6$	$1 - h\epsilon - \epsilon/4$	1
$b_{A_2(h)}$	$1 + i\epsilon$	$1 - h\epsilon - \epsilon/4$	$(1+\theta)^2/\theta$	θ
$b_{E_2(h)}$	$(1+\theta)^2/\theta$	$1 - h\epsilon - \epsilon/4$	$1 - h\epsilon - \epsilon/4$	θ
$b_{G_2(h)}$	$1 + k\epsilon$	$(1+\theta)^2/\theta$	$1 - h\epsilon - \epsilon/4$	θ
$H_2^i(h)$	$1 + i\epsilon$	1	$1 + h\epsilon + \epsilon/6$	θ
$H_2^j(h)$	$1 + j\epsilon$	1	$1 + h\epsilon$	θ
$H_2^k(h)$	$1 + k\epsilon$	$1 + h\epsilon + 3\epsilon/6$	1	θ

Table 8. The points in the clause gadget associated with the clause $e_3(h) = (v_i \vee \neg v_j \vee v_k)$ of the equation $e(h) : v_i + v_j + v_k = 0 \pmod 2$.

Point	c_x	c_y	c_z	Weight
$A_3(h)$	$1 + i\epsilon$	$1 - h\epsilon - 2\epsilon/4$	$1 + h\epsilon + 3\epsilon/6$	1
$B_3(h)$	$1 + k\epsilon$	$1 - h\epsilon - 2\epsilon/4$	$1 + h\epsilon + 3\epsilon/6$	θ
$C_3(h)$	$1 + k\epsilon$	$1 - h\epsilon - 2\epsilon/4$	$1 + h\epsilon + 2\epsilon/6$	1
$D_3(h)$	$1 + \theta$	$1 - h\epsilon - 2\epsilon/4$	$1 + h\epsilon + 2\epsilon/6$	θ
$E_3(h)$	$1 + \theta$	$1 - h\epsilon - 2\epsilon/4$	$1 - h\epsilon - 2\epsilon/4$	1
$F_3(h)$	$1 + \theta$	$1 + h\epsilon + 4\epsilon/6$	$1 - h\epsilon - 2\epsilon/4$	θ
$G_3(h)$	$1 + j\epsilon$	$1 + h\epsilon + 4\epsilon/6$	$1 - h\epsilon - 2\epsilon/4$	1
$b_{A_3(h)}$	$1 + i\epsilon$	$1 - h\epsilon - 2\epsilon/4$	$(1+\theta)^2/\theta$	θ
$b_{E_3(h)}$	$(1+\theta)^2/\theta$	$1 - h\epsilon - 2\epsilon/4$	$1 - h\epsilon - 2\epsilon/4$	θ
$b_{G_3(h)}$	$1 + j\epsilon$	$(1+\theta)^2/\theta$	$1 - h\epsilon - 2\epsilon/4$	θ
$H_3^i(h)$	$1 + i\epsilon$	1	$1 + h\epsilon + 3\epsilon/6$	θ
$H_3^j(h)$	$1 + j\epsilon$	$1 + h\epsilon + 4\epsilon/6$	1	θ
$H_3^k(h)$	$1 + k\epsilon$	1	$1 + h\epsilon + 2\epsilon/6$	θ

Table 9. The points in the clause gadget associated with the clause $e_4(h) = (\neg v_i \vee v_j \vee v_k)$ of the equation $e(h) : v_i + v_j + v_k = 0 \pmod 2$.

Point	c_x	c_y	c_z	Weight
$A_4(h)$	$1 + j\epsilon$	$1 - h\epsilon - 3\epsilon/4$	$1 + h\epsilon + 5\epsilon/6$	1
$B_4(h)$	$1 + k\epsilon$	$1 - h\epsilon - 3\epsilon/4$	$1 + h\epsilon + 5\epsilon/6$	θ
$C_4(h)$	$1 + k\epsilon$	$1 - h\epsilon - 3\epsilon/4$	$1 + h\epsilon + 4\epsilon/6$	1
$D_4(h)$	$1 + \theta$	$1 - h\epsilon - 3\epsilon/4$	$1 + h\epsilon + 4\epsilon/6$	θ
$E_4(h)$	$1 + \theta$	$1 - h\epsilon - 3\epsilon/4$	$1 - h\epsilon - 3\epsilon/4$	1
$F_4(h)$	$1 + \theta$	$1 + h\epsilon + 5\epsilon/6$	$1 - h\epsilon - 3\epsilon/4$	θ
$G_4(h)$	$1 + i\epsilon$	$1 + h\epsilon + 5\epsilon/6$	$1 - h\epsilon - 3\epsilon/4$	1
$b_{A_4(h)}$	$1 + j\epsilon$	$1 - h\epsilon - 3\epsilon/4$	$(1+\theta)^2/\theta$	θ
$b_{E_4(h)}$	$(1+\theta)^2/\theta$	$1 - h\epsilon - 3\epsilon/4$	$1 - h\epsilon - 3\epsilon/4$	θ
$b_{G_4(h)}$	$1 + i\epsilon$	$(1+\theta)^2/\theta$	$1 - h\epsilon - 3\epsilon/4$	θ
$H_4^i(h)$	$1 + i\epsilon$	$1 + h\epsilon + 5\epsilon/6$	1	θ
$H_4^j(h)$	$1 + j\epsilon$	1	$1 + h\epsilon + 5\epsilon/6$	θ
$H_4^k(h)$	$1 + k\epsilon$	1	$1 + h\epsilon + 4\epsilon/6$	θ

C Figure for the revenue-gap

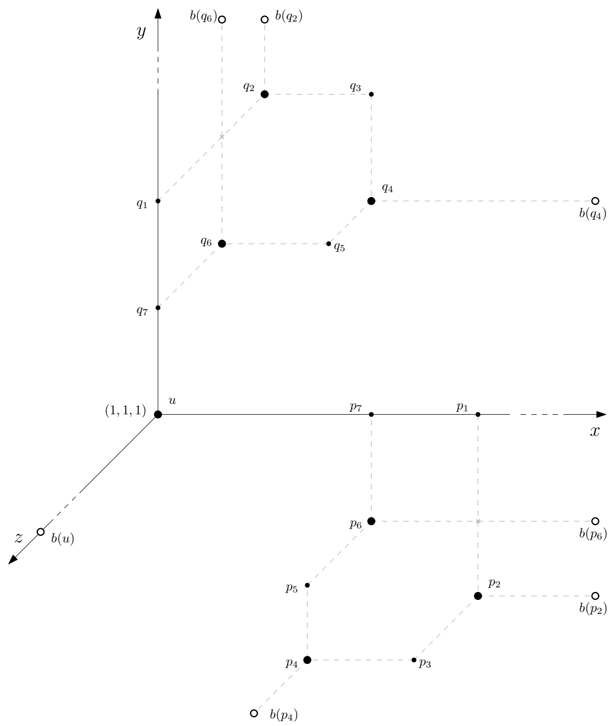

Fig. 3. The instance used in the proof of Theorem 2. Large black disks represent the heavy points of the construction (with weight 1), smaller disks represent light points (with weight θ), and white disks denote blockers. $\theta > 0$ is negligibly small.