Approximating almost all instances of MAX-CUT within a ratio above the Håstad threshold*

A.C. Kaporis† L.M. Kirousis†‡ E. C. Stavropoulos†

August 24, 2006

1 The upper bound

We have n vertices, out of which $n_0 n$ are red, $n_1 n$ are blue and $n_0 + n_1 = 1$. Also, we have bn edges, out of which $b_{00}n$ are red-red, $b_{11}n$ are blue-blue, $b_{01}n = b_{10}n$ are red-blue (that equal blue-red ones) and $b_{01} + b_{00} + b_{11} = b$.

We fix a degree sequence $\langle d_0 n, d_1 n, \ldots, d_{n-1} n\rangle$ where $d_i n$ equals the number of vertices of degree $i = 0, \ldots, d_{n-1}$ such that $\sum_{i=0}^{d_{n-1}} d_i = 1$ and $\frac{1}{2}\sum_{i=0}^{d_{n-1}} i d_i = b$.

We denote as $\delta(0, i, s)$ the scaled number of red colored vertices of degree i and s edges towards blue colored vertices. Similarly we denote $\delta(1, i, s)$. Then for each $i = 0, \ldots, n-1, s = 0, \ldots, i$ it holds:

$$\begin{aligned}
d_i &= \sum_s \left(\delta(0,i,s) + \delta(1,i,s)\right), i = 0, \ldots, n-1, s = 0, \ldots, i. \\
b_{01} &= \sum_{i,s\le i} s\delta(0,i,s), i = 0, \ldots, n-1, s = 0, \ldots, i. \\
b_{10} &= \sum_{i,s\le i} s\delta(1,i,s) = b_{01}, i = 0, \ldots, n-1, s = 0, \ldots, i. \\
b_{00} &= \frac{1}{2}\sum_{i,s\le i} (i-s)\delta(0,i,s), i = 0, \ldots, n-1, s = 0, \ldots, i. \\
b_{11} &= \frac{1}{2}\sum_{i,s\le i} (i-s)\delta(1,i,s), i = 0, \ldots, n-1, s = 0, \ldots, i. \qquad (1)
\end{aligned}$$

We define as

$$N = |\{\langle G, C\rangle : G \text{ is a pairing and } C \text{ a cut satisfying the constraints in (1)}\}| \qquad (2)$$

Then

$$N = \binom{n}{\delta(k,i,s); k = 0,1, i = 0, \ldots, d_{n-1}, 0 \le s \le i} \prod_{i,s} \binom{i}{s}^{\delta(0,i,s)+\delta(1,i,s)} (b_{01}n)!(2b_{00}n)!!(2b_{11}n)!!$$

*This research is partially supported by European Social Fund (ESF), Operational Program for Educational and Vocational Training II (EPEAEK II), and particularly *Pythagoras*. The second author is also partially supported by Future and Emerging Technologies programme of the EU under contract 001907 *"Dynamically Evolving, Large-Scale Information Systems (DELIS)."*

†University of Patras, Department of Computer Engineering and Informatics, GR-265 04 Patras, Greece, email: {kaporis,kirousis, estavrop}@ceid.upatras.gr.

‡Research Academic Computer Technology Institute, P.O. Box 1122, GR-261 10 Patras, Greece.

$$\asymp \prod_{i,s}\left\{\left(\frac{\binom{i}{s}}{\delta(0,i,s)}\right)^{\delta(0,i,s)} \times \left(\frac{\binom{i}{s}}{\delta(1,i,s)}\right)^{\delta(1,i,s)}\right\} \times \frac{b_{01}^{b_{01}} b_{00}^{b_{00}} b_{11}^{b_{11}} 2^{b_{00}+b_{11}}}{e^b} \tag{3}$$

2 Majority cuts

When the b's in (3) above are fixed and constant, we may maximize

$$N = \prod_{k,i,s}\left\{\left(\frac{\binom{i}{s}}{\delta(k,i,s)}\right)^{\delta(k,i,s)}\right\} = \left(\frac{1}{d_0}\right)^{d_0} \prod_{k,i\geq 1,s}\left\{\left(\frac{\binom{i}{s}}{\delta(k,i,s)}\right)^{\delta(k,i,s)}\right\} \tag{4}$$

where in Expression* (4) we have placed into color 0 all $d_0 n$ isolated vertices. In other words, the δ's that rule the 0-degree vertices take the form:

$$\delta(0,0,0) \;=\; d_0, \delta(1,0,0) = 0 \tag{5}$$

- A *majority cut* has all the non-isolated k-colored vertices with at least half of their i stemming semi-edges directed to $(k+1)$-colored vertices. That is,

$$\begin{aligned} \delta(k,i,s) &= 0,\ \forall i \text{ even and } 0 \leq s \leq \frac{i}{2} - 1 \\ \delta(k,i,s) &= 0,\ \forall i \text{ odd and } 0 \leq s \leq \frac{i-1}{2} \end{aligned} \tag{6}$$

- Also a majority cut has no 1-colored vertex of even degree i with exactly $\frac{i}{2}$ stemming semi-edges touching 0-colored vertices. That is, in addition to (6) we also require

$$\delta(1,i,\frac{i}{2}) \;=\; 0,\ \forall i \text{ even} \tag{7}$$

The majority δ's in (6) plus (7) will have to satisfy the b's constraints:

$$\begin{aligned} \Lambda_i : d_i &= \begin{cases} \sum_{s\geq\frac{i}{2}} \delta(0,i,s) + \sum_{s\geq\frac{i}{2}+1} \delta(1,i,s), & i \text{ even} \\ \sum_{s\geq\frac{i+1}{2}} \delta(0,i,s) + \sum_{s\geq\frac{i+1}{2}} \delta(1,i,s), & i \text{ odd} \end{cases} \\ \Lambda_{01} : b_{01} &= \sum_{s\geq\frac{i}{2}} s\delta(0,i=\text{even},s) + \sum_{s\geq\frac{i+1}{2}} s\delta(0,i=\text{odd},s) \\ \Lambda_{10} : b_{10} &= \sum_{s\geq\frac{i}{2}+1} s\delta(1,i=\text{even},s) + \sum_{s\geq\frac{i+1}{2}} s\delta(1,i=\text{odd},s) \\ \Lambda_{00} : b_{00} &= \frac{1}{2}\left(\sum_{s\geq\frac{i}{2}}(i-s)\delta(0,i=\text{even},s) + \sum_{s\geq\frac{i+1}{2}}(i-s)\delta(0,i=\text{odd},s)\right) \\ \Lambda_{11} : b_{11} &= \frac{1}{2}\left(\sum_{s\geq\frac{i}{2}+1}(i-s)\delta(1,i=\text{even},s) + \sum_{s\geq\frac{i+1}{2}}(i-s)\delta(1,i=\text{odd},s)\right) \end{aligned} \tag{8}$$

*It is easy to show the convexity of it with respect to δ's. This is crucial for justifying the uniqueness of the solution of the non-linear system (18). In other words, the local optimum computed via Lagrange multipliers is also a global one.

Taking the derivatives of the logarithm of (4)

$$\frac{\partial \ln N}{\partial \delta(k,i,s)} = \ln\left(\frac{\binom{i}{s}}{\delta(k,i,s)}\right) - 1 = \frac{1}{2}(i-s)\Lambda_{kk} + s\Lambda_{k\bar{k}} + \Lambda_i, k = 0,1 \tag{9}$$

By safely renaming the Lagrange multipliers in (9) we get

$$\frac{\binom{i}{s}}{\delta(k,i,s)} = \Lambda_i \Lambda_{kk}^{-\frac{1}{2}(i-s)} \Lambda_{k\bar{k}}^{-s} \tag{10}$$

We plug (10) into (4) and using the constraints in (8) we can get

$$\begin{aligned}
\prod_{k,i,s}\left\{\left(\frac{\binom{i}{s}}{\delta(k,i,s)}\right)^{\delta(k,i,s)}\right\} &= \prod_{k,i,s}\left\{\left(\Lambda_i \Lambda_{kk}^{-\frac{1}{2}(i-s)} \Lambda_{k\bar{k}}^{-s}\right)^{\delta(k,i,s)}\right\} \\
&= \prod_i \left(\Lambda_i^{\sum_{k,s}\delta(k,i,s)}\right) \prod_{k=0,1}\left(\Lambda_{kk}^{-\frac{1}{2}\sum_{i,s}(i-s)\delta(k,i,s)} \Lambda_{k\bar{k}}^{-\sum_{i,s} s\delta(k,i,s)}\right) \\
&= \prod_i \left(\Lambda_i^{d_i}\right) \prod_{k=0,1}\left(\Lambda_{kk}^{-b_{kk}} \Lambda_{k\bar{k}}^{-b_{k\bar{k}}}\right)
\end{aligned} \tag{11}$$

We can simplify multiplier Λ_i in (11) by manipulating (10) as follows:

$$\begin{aligned}
\Lambda_i \delta(k,i,s) &= \binom{i}{s} \Lambda_{kk}^{\frac{1}{2}(i-s)} \Lambda_{k\bar{k}}^{s} \\
\sum_{k,s} \Lambda_i \delta(k,i,s) &= \begin{cases} \sum_{s\geq\frac{i}{2}} \binom{i}{s}\Lambda_{00}^{\frac{1}{2}(i-s)}\Lambda_{01}^{s} + \sum_{s\geq\frac{i+1}{2}} \binom{i}{s}\Lambda_{11}^{\frac{1}{2}(i-s)}\Lambda_{10}^{s}, & \text{if } i \text{ even} \\ \sum_{s\geq\frac{i+1}{2}} \binom{i}{s}\Lambda_{00}^{\frac{1}{2}(i-s)}\Lambda_{01}^{s} + \sum_{s\geq\frac{i+1}{2}} \binom{i}{s}\Lambda_{11}^{\frac{1}{2}(i-s)}\Lambda_{10}^{s}, & \text{if } i \text{ odd} \end{cases} \\
\Lambda_i d_i &= \begin{cases} \Phi_{i,\lfloor\frac{i+1}{2}\rfloor-1}\left(\Lambda_{00}^{\frac{1}{2}}, \Lambda_{01}\right) + \Phi_{i,\lfloor\frac{i+1}{2}\rfloor}\left(\Lambda_{11}^{\frac{1}{2}}, \Lambda_{10}\right), & i \text{ even} \\ \Phi_{i,\lfloor\frac{i+1}{2}\rfloor-1}\left(\Lambda_{00}^{\frac{1}{2}}, \Lambda_{01}\right) + \Phi_{i,\lfloor\frac{i+1}{2}\rfloor-1}\left(\Lambda_{11}^{\frac{1}{2}}, \Lambda_{10}\right), & i \text{ odd} \end{cases}
\end{aligned} \tag{12}$$

where

$$\Phi_{i,t}(x,y) = (x+y)^i - \sum_{s=0}^{t} \binom{i}{s} y^s x^{i-s} \tag{13}$$

By (12) for majority cuts the objective function (4) becomes:

$$\begin{aligned}
\prod_{k,i,s}\left\{\left(\frac{\binom{i}{s}}{\delta(k,i,s)}\right)^{\delta(k,i,s)}\right\} &= \prod_i \left\{\left(\frac{\Phi_{i,\lfloor\frac{i+1}{2}\rfloor-1}\left(\Lambda_{00}^{\frac{1}{2}}, \Lambda_{01}\right) + \Phi_{i,\lfloor\frac{i+1}{2}\rfloor-I_i}\left(\Lambda_{11}^{\frac{1}{2}}, \Lambda_{10}\right)}{d_i}\right)^{d_i}\right\} \\
&\times \prod_{k=0,1}\left(\Lambda_{kk}^{-b_{kk}} \Lambda_{k\bar{k}}^{-b_{k\bar{k}}}\right) \\
&\text{where } I_i = \begin{cases} 0, & i \text{ even} \\ 1, & i \text{ odd} \end{cases}
\end{aligned} \tag{14}$$

Expression (14) is implicit functions of the 4-tuple of the Lagrange multipliers

$$\mathcal{L} = \langle \Lambda_{01}, \Lambda_{10}, \Lambda_{00}, \Lambda_{11} \rangle \tag{15}$$

We can derive implicitly the values of (15) as follows. For majority cuts, by (12) expression (10) can be written:

$$\delta(k,i,s) = \frac{d_i}{\Phi_{i,\lfloor\frac{i+1}{2}\rfloor-1}\left(\Lambda_{00}^{\frac{1}{2}}, \Lambda_{01}\right) + \Phi_{i,\lfloor\frac{i+1}{2}\rfloor-I_i}\left(\Lambda_{11}^{\frac{1}{2}}, \Lambda_{10}\right)} \binom{i}{s} \Lambda_{kk}^{\frac{1}{2}(i-s)} \Lambda_{k\bar{k}}^{s} \tag{16}$$

Majority 4×4**.** For the majority case, taking appropriate summations of the δ's in (16) and using the constraints (8) we can express each 4-tuple of given b's

$$\mathcal{B}(b) = \langle b_{01}, b_{10}, b_{00}, b_{11} \rangle \tag{17}$$

as functions of the Λ's in (15).

$$\begin{aligned}
b_{01} &= \Lambda_{01} \times \sum_{i=1}^{d_{\max}} \left(d_i \frac{\left\{\Phi_{i,\lfloor\frac{i+1}{2}\rfloor-1}\left(\Lambda_{00}^{\frac{1}{2}}, \Lambda_{01}\right)\right\}'_{\Lambda_{01}}}{\Phi_{i,\lfloor\frac{i+1}{2}\rfloor-1}\left(\Lambda_{00}^{\frac{1}{2}}, \Lambda_{01}\right) + \Phi_{i,\lfloor\frac{i+1}{2}\rfloor-I_i}\left(\Lambda_{11}^{\frac{1}{2}}, \Lambda_{10}\right)} \right) \\
b_{10} &= \Lambda_{10} \times \sum_{i=1}^{d_{\max}} \left(d_i \frac{\left\{\Phi_{i,\lfloor\frac{i+1}{2}\rfloor-I_i}\left(\Lambda_{11}^{\frac{1}{2}}, \Lambda_{10}\right)\right\}'_{\Lambda_{01}}}{\Phi_{i,\lfloor\frac{i+1}{2}\rfloor-1}\left(\Lambda_{00}^{\frac{1}{2}}, \Lambda_{01}\right) + \Phi_{i,\lfloor\frac{i+1}{2}\rfloor-I_i}\left(\Lambda_{11}^{\frac{1}{2}}, \Lambda_{10}\right)} \right) \\
b_{00} &= \Lambda_{00} \times \sum_{i=1}^{d_{\max}} \left(d_i \frac{\left\{\Phi_{i,\lfloor\frac{i+1}{2}\rfloor-1}\left(\Lambda_{00}^{\frac{1}{2}}, \Lambda_{01}\right)\right\}'_{\Lambda_{00}}}{\Phi_{i,\lfloor\frac{i+1}{2}\rfloor-1}\left(\Lambda_{00}^{\frac{1}{2}}, \Lambda_{01}\right) + \Phi_{i,\lfloor\frac{i+1}{2}\rfloor-I_i}\left(\Lambda_{11}^{\frac{1}{2}}, \Lambda_{10}\right)} \right) \\
b_{11} &= \Lambda_{11} \times \sum_{i=1}^{d_{\max}} \left(d_i \frac{\left\{\Phi_{i,\lfloor\frac{i+1}{2}\rfloor-I_i}\left(\Lambda_{11}^{\frac{1}{2}}, \Lambda_{10}\right)\right\}'_{\Lambda_{11}}}{\Phi_{i,\lfloor\frac{i+1}{2}\rfloor-1}\left(\Lambda_{00}^{\frac{1}{2}}, \Lambda_{01}\right) + \Phi_{i,\lfloor\frac{i+1}{2}\rfloor-I_i}\left(\Lambda_{11}^{\frac{1}{2}}, \Lambda_{10}\right)} \right)
\end{aligned} \tag{18}$$

2.1 Optimization via a 4×4 system

The objective function (3) now becomes:

$$\begin{aligned}
f(\mathcal{B}(b)) &= \left(\frac{1}{2b}\right)^b \prod_i \left\{ \Phi_{i,\lfloor\frac{i+1}{2}\rfloor-1}\left(\Lambda_{00}^{\frac{1}{2}}, \Lambda_{01}\right) + \Phi_{i,\lfloor\frac{i+1}{2}\rfloor-I_i}\left(\Lambda_{11}^{\frac{1}{2}}, \Lambda_{10}\right) \right\}^{d_i} \\
&\quad \times \left(\frac{b_{01}}{\Lambda_{01}}\right)^{b_{01}} \left(\frac{b_{00}}{\Lambda_{00}}\right)^{b_{00}} \left(\frac{b_{11}}{\Lambda_{11}}\right)^{b_{11}} \left(\frac{1}{\Lambda_{10}}\right)^{b_{10}} 2^{b_{00}+b_{11}}
\end{aligned} \tag{19}$$

where for each fixed tuple of b's as in (17) the corresponding Λ's are computed by the numeric solution of system (18).

Optimization target. Given is a fixed constant b which denoted the density of the random graph. The target is to compute the *minimum* $b_{01} = b_{01}(b)$ such that for all 4-tuples

$$\mathcal{B}(b) = \langle b_{01}, b_{10}, b_{00}, b_{11} \rangle \text{ with } b_{10} = b_{01} \text{ and } \frac{b_{01}}{2} + \frac{b_{10}}{2} + b_{00} + b_{11} = b \tag{20}$$

the *majority* function in (19) remains < 1.