Energy-Efficient Wireless Network Design*

Ioannis Caragiannis Christos Kaklamanis Panagiotis Kanellopoulos

Research Academic Computer Technology Institute and
Department of Computer Engineering and Informatics
University of Patras, 26500 Rio, Greece
E–mail: {caragian,kakl,kanellop}@ceid.upatras.gr

Abstract

A crucial issue in *wireless networks* is to efficiently support communication patterns that are typical in traditional (wired) networks. These include broadcasting, multicasting, and gossiping (all-to-all communication). In this work, we study such problems in *static ad hoc networks*. Since, in ad hoc networks energy is a scarce resource, the important engineering question to be solved is to guarantee a desired communication pattern minimizing the total energy consumption. Motivated by this question, we study a series of *wireless network design problems* and present new approximation algorithms and inapproximability results.

1 Introduction

Wireless networks have received significant attention during the recent years. Especially, *ad hoc networks* emerged due to their potential applications in battlefield, emergency disaster relief, etc. [18, 21]. Unlike traditional wired networks or cellular wireless networks, no wired backbone infrastructure is installed for ad hoc networks.

A node (or station) in these networks is equipped with an omnidirectional antenna which is responsible for sending and receiving signals. Communication is established by assigning to each station a transmitting power. In the most common power attenuation model [18], the signal power falls as $1/r^{\alpha}$, where r is the distance from the transmitter and α is a constant which depends on the wireless environment (typical values of α are between 1 and 6). So, a transmitter can send a signal to a receiver if $\frac{P_s}{d(s,t)^{\alpha}} \geq \gamma$ where P_s is the power of the transmitting signal, $d(s,t)$ is the Euclidean distance between the transmitter and the receiver, and γ is the receiver's power threshold for signal detection which is usually normalized to 1.

So, communication from a node s to another node t may be established either directly if the two nodes are close enough and s uses adequate transmitting power, or by using intermediate nodes. Observe that due to the nonlinear power attenuation, relaying the signal between intermediate nodes may result in energy conservation.

A crucial issue in ad hoc networks is to support communication patterns that are typical in traditional networks. These include broadcasting, multicasting, and gossiping (all–to–all communication). Since establishing a communication pattern strongly depends on the use of energy, the important engineering question to be solved is to guarantee a desired communication pattern minimizing the

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total energy consumption. In this work, we consider a series of wireless network design problems in static ad hoc networks which we formulate below.

Formulation of problems studied. We model a static ad hoc network by a complete directed graph $G = (V, E)$, where $|V| = n$, with a non-negative edge cost function $c : E \to R^+$. Given a non-negative node weight assignment $w : V \to R^+$, the *transmission graph* G_w is the directed graph defined as follows. It has the same set of nodes as G and a directed edge (u, v) belongs to G_w if the weight assigned to node u is at least the cost of the edge (u, v), i.e., $w(u) \geq c(u, v)$. Intuitively, the weight assignment corresponds to the energy levels at which each node operates (i.e., transmits messages) while the cost between two nodes indicates the minimum energy level necessary to send messages from one node to the other. Usually, the edge cost function is symmetric (i.e., $c(u, v) = c(v, u)$). Asymmetric edge cost functions can be used to model medium abnormalities or batteries with different energy levels [17].

The problems we study in this paper can be stated as follows. Given a complete directed graph $G = (V, E)$, where $|V| = n$, with non-negative edge costs $c : E \to R^+$, find a non-negative node weight assignment $w : V \to R^+$ such that the transmission graph G_w maintains a connectivity property and the sum of weights is minimized. Such a property is defined by a requirement matrix $R = (r_{ij}) \in \{0, 1\}$ where r_{ij} is the number of directed paths required in the transmission graph from node v_i to node v_j. Depending on the connectivity property for the transmission graph, we may define the following problems.

In Minimum Energy Steiner Subgraph (MESS), the requirement matrix is symmetric. Alternatively, we may define the problem by a set of nodes $D \subseteq V$ partitioned into p disjoint subsets $D_1, D_2, ..., D_p$. The entries of the requirement matrix are now defined as $r_{ij} = 1$ if $v_i, v_j \in D_k$ for some k and $r_{ij} = 0$, otherwise. The Minimum Energy Subset Strongly Connected Subgraph (MESSCS) is the special case of MESS with $p = 1$ while the Minimum Energy Strongly Connected Subgraph (MESCS) is the special case of MESSCS with $D = V$ (i.e., the transmission graph is required to span all nodes of V and to be strongly connected). The authors of [1] and [4] study MESCS under the extra requirement that the transmission graph contains a bidirected subgraph (i.e., a directed graph in which the existence of a directed edge implies that its opposite directed edge also exists in the graph), which maintains the connectivity requirements of MESCS. By adding this extra requirement to MESS and MESSCS, we obtain the bidirected MESS and bidirected MESSCS, respectively. I.e., the requirement for the transmission graph in bidirected MESS (resp., bidirected MESSCS) is to contain as a subgraph a bidirected graph satisfying the connectivity requirements of MESS (resp., MESSCS).

In Minimum Energy Multicast Tree (MEMT), the connectivity property is defined by a root node v_0 and a set of nodes $D \subseteq V - \{v_0\}$ such that $r_{ij} = 1$ if $i = 0$ and $v_j \in D$ and $r_{ij} = 0$, otherwise. The Minimum Energy Broadcast Tree (MEBT) is the special case of MEMT with $D = V - \{v_0\}$. By inverting the connectivity requirements, we obtain the following two problems: the Minimum Energy Inverse Multicast Tree (MEIMT) where the connectivity property is defined by a root node v_0 and a set of nodes $D \subseteq V - \{v_0\}$ such that $r_{ij} = 1$ if $v_i \in D$ and $j = 0$ and $r_{ij} = 0$, otherwise, and the Minimum Energy Inverse Broadcast Tree (MEIBT) which is the special case of MEIMT with $D = V - \{v_0\}$.

Table 1 summarizes abbreviations used in the paper for the problems studied, as well as for other combinatorial problems used several times in the proofs.

Previous work. For bidirected MESCS, [1] shows a constant approximation algorithm in symmetric graphs and a logarithmic inapproximability result in asymmetric graphs. The same reduction used to prove this inapproximability result can be used for proving a logarithmic inapproximability

Abbreviation	Problem
MESS	Minimum Energy Steiner Subgraph
MESSCS	Minimum Energy Subset Strongly Connected Subgraph
MESCS	Minimum Energy Strongly Connected Subgraph
MEMT	Minimum Energy Multicast Tree
MEBT	Minimum Energy Broadcast Tree
MEIMT	Minimum Energy Inverse Multicast Tree
MEIBT	Minimum Energy Inverse Broadcast Tree
SF	Steiner Forest
ST	Steiner Tree
DST	Directed Steiner Tree
MSA	Minimum Spanning Arborescence
NWSF	Node-Weighted Steiner Forest
NWST	Node-Weighted Steiner Tree

Table 1: Abbreviations for problems used in the paper.

result for MESCS in asymmetric graphs as well. MESCS in symmetric graphs is studied by Kirousis et al. [15] who present (among other results) a 2-approximation algorithm. Clementi et al. [9] study geometric versions of the problem and show approximation-preserving reductions from Vertex Cover on bounded-degree graphs to geometric instances of MESCS. By adapting the reduction of [9] and using the hardness results of [2], we can obtain an inapproximability factor of 313/312 for MESCS in symmetric graphs. As observed in [1], this result holds for bidirected MESCS as well.

Liang [17] shows an $O(|D|^\epsilon)$– and an $O(n^\epsilon)$–approximation algorithm for MEMT and MEBT, respectively, for any constant $\epsilon > 0$. These results follow by using an intuitive reduction of any instance of MEMT to an instance of Directed Steiner Tree and then applying the algorithm of Charikar et al. [6] for computing an approximate directed Steiner tree which gives an approximate solution to MEMT. Note that the work of Liang does not answer the question whether MEMT and MEBT are strictly easier to approximate than Directed Steiner Tree or not. For MEMT in symmetric graphs, Liang [17] shows an $O(\ln^3|D|)$-approximation algorithm while, for MEBT in symmetric graphs, we show in [5] a $10.8\ln n$-approximation algorithm by reducing instances of the problem to instances of Node-Weighted Connected Dominating Set and using the algorithm of [12] for computing an approximate connected dominating set which gives an approximate solution to MEBT. Constant approximation algorithms for geometric versions of MEBT are presented in [8, 20].

Related combinatorial problems. In the paper, we usually refer to classical combinatorial optimization problems. For completeness, we present their definitions here. The Steiner Forest (SF) problem is defined as follows. Given an undirected graph $G = (V, E)$ with an edge cost function $c : E \to R^+$ and a set of nodes $D \subseteq V$ partitioned into p disjoint sets $D_1, ..., D_p$, compute a subgraph H of G of minimum total edge cost such that any two nodes v_i, v_j belonging to the same set D_k for some k are connected through a path in H. Steiner Tree (ST) is the special case of SF with $p = 1$. An instance of the Directed Steiner Tree (DST) is defined by a directed graph $G = (V, E)$ with an edge cost function $c : E \to R^+$, a root node $v_0 \in V$, and a set of terminals $D \subseteq V - \{v_0\}$. Its objective is to compute a tree of minimum edge cost which is directed out of v_0 and spans all nodes of D. The special case of DST, with $D = V - \{v_0\}$ is called Minimum Spanning Arborescence (MSA). The Node–Weighted Steiner Forest (NWSF) problem is defined as follows. Given an undirected graph $G = (V, E)$ with a node cost function $c : V \to R^+$ and a set of nodes $D \subseteq V$ partitioned into p disjoint sets $D_1, ..., D_p$, compute a subgraph H of G of minimum total node cost such that any two nodes v_i, v_j belonging to the same set D_k for some k are connected through a path in H. Node–Weighted Steiner Tree (NWST) is the special case of NWSF with $p = 1$.

Our results. In the rest of this section we give an overview of our results. The best known results for the problems studied (including the results in this paper) are summarized in Table 2.

	Approximability in asymmetric graphs		Approximability in symmetric graphs	
Problem	Lower Bound	Upper Bound	Lower Bound	Upper Bound
MESS	$\Omega(\ln^{2-\epsilon} n)$ *		313/312 [9]	4 *
MESSCS	$\Omega(\ln^{2-\epsilon} n)$ *		313/312 [9]	3.1 *
MESCS	$\Omega(\log n)$ [3]	$O(\log n)$ *, [3]	313/312 [9]	2 [15]
bidirected MESS	$\Omega(\log n)$ [1]	$1.61 \ln \lvert D\rvert$ *	96/95 *	4 *
bidirected MESSCS	$\Omega(\log n)$ [1]	$1.35 \ln \lvert D\rvert$ *	96/95 *	3.1 *
bidirected MESCS	$\Omega(\log n)$ [1]	$1.35 \ln n$ *	313/312 [9]	5/3 [1]
MEMT	$\Omega(\ln^{2-\epsilon} n)$ *	$O(\lvert D\rvert^{\epsilon})$ [17]	$\Omega(\log n)$ [8]	$O(\log n)$ *
MEBT	$\Omega(\log n)$ [8]	$O(\log n)$ *, [3]	$\Omega(\log n)$ [8]	$O(\log n)$ *, [3, 5]
MEIMT	$\Omega(\ln^{2-\epsilon} n)$ *	$O(\lvert D\rvert^{\epsilon})$ *	96/95 *	1.55 *
MEIBT	1 *	1 *	1 *	1 *

Table 2: The best known results for the problems studied. (*) denotes results presented in this paper.

In Section 2, we present constant approximation algorithms for MESS and MESSCS in symmetric graphs (i.e., input instances with symmetric edge cost functions) exploiting known efficient approximation algorithms for SF and ST. These results also apply to bidirected MESS and bidirected MESSCS in symmetric graphs. For bidirected MESSCS, we also give an inapproximability result using an approximation-preserving reduction from ST.

We also present approximation algorithms for MEMT and MEBT in symmetric graphs (Section 3) with logarithmic approximation ratios by using a new reduction of instances of MEMT and MEBT to instances of NWST, a problem which is known to be approximable within a logarithmic factor [12]. These results are asymptotically optimal since MEBT in symmetric graphs has been proved to be inapproximable within a sublogarithmic factor [8]. Our result for MEMT improves the polylogarithmic approximation algorithm of Liang [17] while the result for MEBT improves the result of [5] by a multiplicative factor of 4.

In Section 4, we observe that MEIBT is equivalent to MSA and, thus, it can be solved in polynomial time in both symmetric and asymmetric graphs. For MEIMT, we observe that, in symmetric graphs, it is equivalent to ST (and, thus, it can be approximated within a constant factor) and that, in asymmetric graphs, it is equivalent to DST.

We also show that, in asymmetric graphs, MEMT and MESSCS are at least as hard to approximate as DST. Using a recent inapproximability result for DST due to Halperin and Krauthgamer [14] we obtain polylogarithmic inapproximability results for these problems. On the positive side, we show that MESSCS in asymmetric graphs can be solved by solving an instance of MEMT and an instance of MEIMT. By exploiting the equivalence of MEIMT to DST, using the reduction of Liang [17] for MEMT to DST and applying the approximation algorithm of [6], we obtain an $O(|D|^{\epsilon})$-approximation algorithm, for any $\epsilon > 0$. These results are presented in Section 5.

For MEBT in asymmetric graphs, we present an $O(\ln n)$-approximation algorithm exploiting a recent result of Zosin and Khuller [22]. This result is asymptotically optimal and significantly improves the $O(n^{\epsilon})$-approximation algorithm due to Liang [17]. We also show that MESCS in asymmetric graphs can be solved by solving an instance of MEBT and an instance of MEIBT. Using the logarithmic approximation algorithm for MEBT and the fact that MEIBT can be solved optimally in polynomial time, we obtain an asymptotically optimal logarithmic approximation algorithm for MESCS in asymmetric graphs. These results are presented in Section 6. Independently, Calinescu et al. ([3]) achieve a similar approximation bound by a simpler algorithm which constructs a tree incrementally, using sophisticated set-covering techniques for the analysis.

In Section 7 we present an $O(\ln |D|)$-approximation algorithm for bidirected MESS in asymmetric graphs by using a new reduction of instances of the problem to instances of NWSF, a problem which is known to be approximable within a logarithmic factor [12]. As corollaries, we obtain $O(\ln |D|)$– and $O(\ln n)$–approximation algorithms for bidirected MESSCS and bidirected MESCS, respectively. These results asymptotically match the inapproximability result for bidirected MESCS of Althaus et al. [1]. A slightly inferior logarithmic approximation bound for bidirected MESCS has been independently obtained in [3] using different techniques.

2 Symmetric graphs and connectivity requirements

In this section, we first show constant approximation algorithms for MESS, MESSCS, bidirected MESS, and bidirected MESSCS, extending the algorithm of Kirousis et al. [15] for MESCS in symmetric graphs. Then, we show an inapproximability result for bidirected MESSCS.

Consider an instance I_{MESS} of MESS which consists of a complete directed graph $G = (V, E)$, a symmetric edge cost function $c : E \to R^+$ and a set of terminals $D \subseteq V$ partitioned into p disjoint subsets $D_1, ..., D_p$. We construct the instance I_{SF} of SF which consists of the complete undirected graph $H = (V, E')$, the edge cost function $c' : E' \to R^+$ defined as $c'(u, v) = c(u, v) = c(v, u)$ on the undirected edges of E', and the set of terminals D together with its partition into the sets $D_1, ..., D_p$. Consider a solution for I_{SF} that consists of a subgraph $F = (V, A)$ of H. We construct the weight assignment w to the nodes of V by setting $w(u) = 0$ if there is no edge touching u in A and $w(u) = \max_{v:(u,v)\in A}\{c'(u, v)\}$, otherwise. An example of this construction is presented in Figure 1. We show the following lemma.

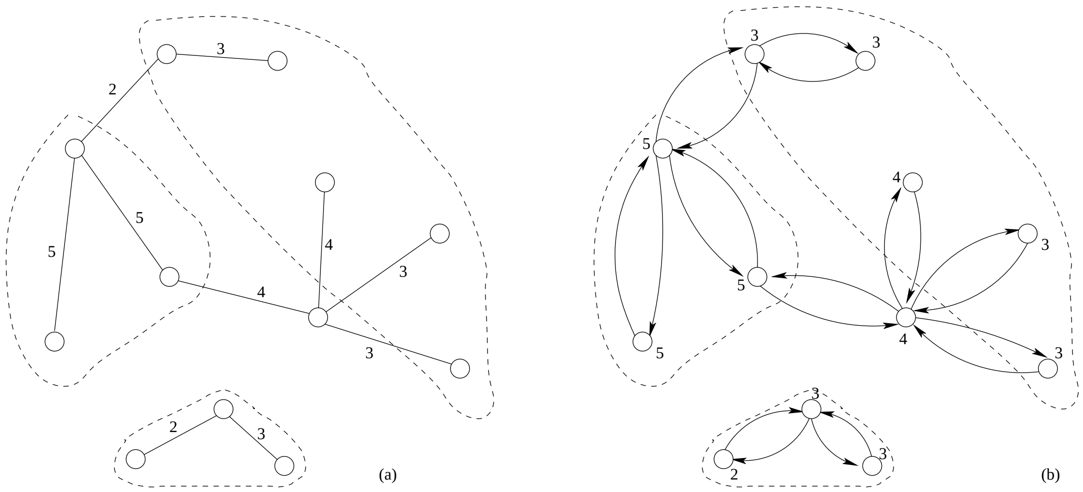

Figure 1: Transforming a solution for I_{SF} (a) to a solution for I_{MESS} (b). In (a), numbers on the edges of the solution denote their cost. In (b), numbers are associated with the nodes and denote their weight. In both cases, the dashed closed lines indicate the subsets in which the set of terminals is partitioned.

Lemma 1 *If F is a ρ–approximate solution for I_{SF} then w is a 2ρ–approximate solution for I_{MESS}.*

Proof: First, we observe that w is indeed a solution for I_{MESS}. The transmission graph G_w contains two opposite directed edges (u, v) and (v, u) for each undirected edge of A. This means that for any path between two nodes u and v in F, there exists a directed path from u to v and a directed path from v to u in G_w. Thus, since $F = (V, A)$ maintains the connectivity requirements of SF for instance I_{SF}, the transmission graph G_w maintains the connectivity requirements of MESS for instance I_{MESS}.

Now, assume that F is a ρ–approximate solution for I_{SF}, i.e., $COST(I_{SF}) \leq \rho \cdot OPT(I_{SF})$, where $COST(I_{SF})$ is the cost of the solution F and $OPT(I_{SF})$ is the cost of the optimal solution of SF for I_{SF}. We will show that the cost $COST(I_{MESS})$ of the solution w is upper-bounded by $2 \cdot COST(I_{SF})$ and, furthermore, that the cost of the optimal solution of SF for I_{SF} is upper-bounded by the cost $OPT(I_{MESS})$ of the optimal solution of MESS for I_{MESS}. In this way, we will obtain that w is a 2ρ–approximate solution for I_{MESS} since

$$COST(I_{MESS}) \leq 2 \cdot COST(I_{SF}) \leq 2\rho \cdot OPT(I_{SF}) \leq 2\rho \cdot OPT(I_{MESS}).$$

Indeed, the cost of the solution w is

$$\begin{aligned} COST(I_{MESS}) &= \sum_{u \in V} \max_{v:(u,v) \in A} \{c'(u, v)\} \\ &\leq \sum_{u \in V} \sum_{v:(u,v) \in A} c'(u, v) \\ &= 2 \sum_{(u,v) \in A} c'(u, v) \\ &= 2 \cdot COST(I_{SF}). \end{aligned}$$

Next, in order to show that $OPT(I_{SF})$ is upper-bounded by $OPT(I_{MESS})$, we will construct a solution $F' = (V, A')$ of SF for I_{SF} of cost at most $OPT(I_{MESS})$. Let w' be an optimal solution for I_{MESS} and let $G_{w'}$ be the corresponding transmission graph.

We first compute a subgraph G' of $G_{w'}$ that maintains the connectivity requirements of MESS for I_{MESS} and in which every non–trivial connected component (i.e., every connected component that is not an isolated node) is strongly connected. This is done as follows. Initially, G' contains all nodes of V but no edges. We satisfy all connectivity requirements of I_{MESS} in steps by adding edges of $G_{w'}$ to G'. In each step, in order to satisfy the connectivity requirements between two nodes v_1 and v_2 (if they are not already satisfied), we find a directed path from v_1 to v_2 and a directed path from v_2 to v_1 in $G_{w'}$ and we add the edges of the two paths into G'. We repeat this procedure for each unsatisfied connectivity requirement of I_{MESS}.

We use induction to prove that, in each step, the non-trivial connected components of G' are strongly connected. Denote by u and v the nodes which are connected in the first step. These are connected through a directed path p_1 from u to v and a directed path p_2 from v to u. Clearly, nodes which are not on the paths p_1 and p_2 are still isolated. Also, v is reached by each node in p_1 (including u) through a subpath of p_1 and reaches each node in p_2 (including u) through a subpath of p_2. Similarly, u is reached by each node in p_2 through a subpath of p_2 and reaches each node in p_1 through a subpath of p_1. Hence, each pair of nodes in the union of paths p_1 and p_2 (which may include u and/or v) is connected through directed paths in both directions. This means that the connected component containing u and v is strongly connected. Now, assume that, after the i–th step, we have satisfied the first i connectivity requirements and that each non-trivial connected component of G' is strongly connected. In the $(i + 1)$–th step, we add edges to G' so that the connectivity requirements between two nodes u' and v' are satisfied. Note that u' and v' belong to different connected components (including trivial ones) at the end of the i–th step, otherwise the connectivity requirement between them would be already satisfied. We connect u' to v' through a directed path p'_1 and v' to u' by a directed path p'_2. This does not affect connected components having no node in p'_1 and p'_2; such connected components are still either strongly connected or isolated nodes after the $(i+1)$–th step. Now, consider the connected component connecting u' and v' obtained after adding the edges in p'_1 and p'_2 to G'. Node v' is reached by each node in p'_1 through a subpath of p'_1 and, consequently, it is reached by each node in connected components in G' having a node in p'_1. It also reaches each node in p'_2 through a subpath of p'_2 and, consequently, it reaches each node in connected components of G' having a node in p'_2. Similarly, u is reached by each node in connected components of G' having a node in p'_2 and reaches each node in connected components of G' having a node in p'_1. Hence, for each pair of nodes in connected components in G' containing a node in paths p'_1 or p'_2, there exist directed paths connecting them in both directions. This means that the connected component containing u' and v' is strongly connected.

Let G'_1, ..., G'_k be the non–trivial connected components of G'. In each graph G'_i, arbitrarily pick a node r_i and find a tree T_i directed towards r_i and spanning all nodes of G'_i. This can be done since G'_i is strongly connected. Let $T = (V, E(T))$ be the graph defined by the union of these trees. We construct the solution $F' = (V, A')$ for I_{SF} as follows. For every directed edge (u, v) of $E(T)$, A' contains the (undirected) edge (u, v). Observe that, for $i = 1, ..., p$, the terminals of D_i belong to the same connected component G'_i. Since from each of the terminals of D_i there exists a directed path to node r_i in T, any two terminals of D_i are connected through a path in F'. Thus, our construction guarantees that F' is indeed a solution to I_{SF}. Its cost is

$$
\begin{aligned}
\sum_{(u,v)\in A'} c'(u,v) &= \sum_{(u,v)\in E(T)} c(u,v) \\
&\leq \sum_{u\in V} w(u) \\
&= OPT(I_{MESS})
\end{aligned}
$$

where the inequality follows by the fact that each node of T has out–degree at most 1. ■

We can solve I_{SF} using the 2-approximation algorithm of Goemans and Williamson [11] for SF. When $p = 1$ (i.e., when I_{MESS} is actually an instance of MESSCS), the instance I_{SF} is actually an instance of ST which can be approximated within $1 + \frac{1}{2}\ln 3 \approx 1.55$ using an algorithm of Robins and Zelikovsky [19]. We obtain the following result.

Theorem 2 *There exist a* 4- *and a* 3.1-*approximation algorithm for MESS and MESSCS in symmetric graphs, respectively.*

Note that the transmission graph constructed by the technique in this section contains a bidirected subgraph that maintains the connectivity requirements of MESS and, thus, our algorithms for MESS and MESSCS provide solutions to bidirected MESS and bidirected MESSCS, respectively. The analysis still holds if we consider instances of bidirected MESS and bidirected MESSCS instead of instances of MESS and MESSCS, respectively. Thus, the approximation guarantees of Theorem 2 hold for bidirected MESS and bidirected MESSCS in symmetric graphs as well.

The inapproximability result of 313/312 for MESCS mentioned in the introduction holds for bidirected MESCS in symmetric graphs as well, and, hence, it holds for bidirected MESSCS. We now show a simple approximation-preserving reduction from ST to bidirected MESSCS in symmetric graphs.

Given an instance I_{ST} of ST which consists of an undirected graph $G = (V, E)$ with edge cost function $c : E \rightarrow R^+$, and a set of terminals $D \subseteq V$, construct the instance $I_{bMESSCS}$ as follows.

$I_{bMESSCS}$ consists of a complete directed graph $H = (U, A)$ with symmetric edge cost function $c' : A \to R^+$, and a set of terminals $D' \subseteq U$. The set of nodes U contains a node h_v for each node v of V and two nodes $h_{(u,v)}$ and $h_{(v,u)}$ for each edge (u, v) of E. The edge cost function c' is defined as $c'(h_u, h_{(u,v)}) = c'(h_{(u,v)}, h_u) = c'(h_{(v,u)}, h_v) = c'(h_v, h_{(v,u)}) = 0$ and $c'(h_{(u,v)}, h_{(v,u)}) = c'(h_{(v,u)}, h_{(u,v)}) = c(u, v)$, for each edge (u, v) of E, while all other directed edges of A have infinite cost. The construction is presented in Figure 2. The set of terminals is defined as $D' = \{h_u \in U | u \in D\}$. We show the following lemma.

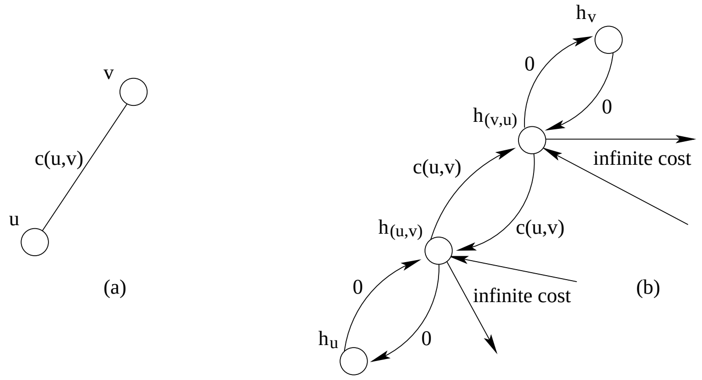

Figure 2: An edge in I_{ST} (a) and the corresponding structure in $I_{bMESSCS}$ (b). In (b), all edges which are incident to $h_{(u,v)}$ but are not incident to either $h_{(v,u)}$ or h_u, as well as all edges which are incident to $h_{(v,u)}$ but are not incident to either $h_{(u,v)}$ or h_v have infinite cost.

Lemma 3 *A ρ-approximate solution for $I_{bMESSCS}$ reduces in polynomial time to a ρ-approximate solution for I_{ST}.*

Proof: Consider a solution w to $I_{bMESSCS}$, i.e., a weight assignment to the nodes of U such that the transmission graph H_w contains a bidirected subgraph which maintains the connectivity requirements of $I_{bMESSCS}$. Let H' be the bidirected subgraph of H_w which consists of all the bidirected edges of H_w. Clearly, this graph maintains the connectivity requirements for $I_{bMESSCS}$.

We construct a solution $F = (V, E')$ to I_{ST} as follows. For every pair of nodes $h_{(u,v)}, h_{(v,u)}$ having both $w(h_{(u,v)}) \geq c'(h_{(u,v)}, h_{(v,u)})$ and $w(h_{(v,u)}) \geq c'(h_{(v,u)}, h_{(u,v)})$, we add the edge (u, v) to E', i.e., we add the undirected edge between nodes u and v in E' only if there are two opposite directed paths between nodes h_u and h_v in H' containing the nodes $h_{(u,v)}$ and $h_{(v,u)}$. Since H' maintains the connectivity requirements for $I_{bMESSCS}$, F is a feasible solution for I_{ST}.

Now, assume that w is a ρ–approximate solution for $I_{bMESSCS}$, i.e., $COST(I_{bMESSCS}) \leq \rho \cdot OPT(I_{bMESSCS})$, where $COST(I_{bMESSCS})$ is the cost of the solution w and $OPT(I_{bMESSCS})$ is the cost of the optimal solution of bidirected MESSCS for $I_{bMESSCS}$. We will show that $COST(I_{bMESSCS})$ is lower-bounded by $2 \cdot COST(I_{ST})$, where $COST(I_{ST})$ denotes the cost of F and, furthermore, that $OPT(I_{bMESSCS})$ is upper-bounded by $2 \cdot OPT(I_{ST})$, where $OPT(I_{ST})$ denotes the cost of the optimal solution of ST for I_{ST}. In this way, we will obtain that F is a ρ–approximate solution for I_{ST} since

$$COST(I_{ST}) \leq \frac{1}{2} \cdot COST(I_{bMESSCS}) \leq \frac{\rho}{2} \cdot OPT(I_{bMESSCS}) \leq \rho \cdot OPT(I_{ST}).$$

Indeed, the cost of the solution w is

$$COST(I_{bMESSCS}) \quad = \quad \sum_{y \in U} w(y)$$

$$
\begin{aligned}
&= \sum_{u\in V} w(h_u) + \sum_{(u,v)\in E} \left[w(h_{(u,v)}) + w(h_{(v,u)})\right] \\
&\geq \sum_{(u,v)\in E'} \left[c'(h_{(u,v)}, h_{(v,u)}) + c'(h_{(v,u)}, h_{(u,v)})\right] \\
&= 2 \sum_{(u,v)\in E'} c(u,v) \\
&= 2 \cdot COST(I_{ST})
\end{aligned}
$$

Next, we show that $OPT(I_{bMESSCS})$ is at most $2 \cdot OPT(I_{ST})$. In order to do this, starting from an optimal solution $F' = (V, E'')$ to I_{ST}, we construct a solution w' to $I_{bMESSCS}$ and show that its cost is at most twice the cost of F'. For each edge $(u, v) \in E''$, we assign weight $w'(h_{(u,v)}) = w'(h_{(v,u)}) = c(u, v)$ to nodes $h_{(u,v)}$ and $h_{(v,u)}$. All other nodes in U are assigned zero weight. Consider the bidirected graph H' on the set of nodes U which, for every edge $(u, v) \in E''$, consists of the two opposite directed edges between nodes h_u and $h_{(u,v)}$, the two opposite directed edges between nodes $h_{(u,v)}$ and $h_{(v,u)}$, and the two opposite directed edges between nodes $h_{(v,u)}$ and h_v. Since the graph F' is a solution for I_{ST}, it contains a path p connecting any pair of terminals $u, v \in D$. For each edge (u', v') of p, the bidirected graph H' contains two opposite directed paths of length 3 connecting nodes $h_{u'}$ and $h_{v'}$. Hence, for any pair of terminals $h_u, h_v \in D'$, H' contains two opposite directed paths connecting h_u and h_v in both directions, i.e., it satisfies the connectivity requirements of $I_{bMESSCS}$. Furthermore, recall that the transmission graph $H_{w'}$ contains H' as a subgraph. Therefore w' is a feasible solution for $I_{bMESSCS}$. Thus, the cost of the optimal solution for $I_{bMESSCS}$ is

$$
\begin{aligned}
OPT(I_{bMESSCS}) &\leq \sum_{y\in U} w'(y) \\
&= \sum_{u\in V} w'(h_u) + \sum_{(u,v)\in E} \left[w'(h_{(u,v)}) + w'(h_{(v,u)})\right] \\
&= 2 \sum_{(u,v)\in E''} c(u,v) \\
&= 2 \cdot OPT(I_{ST})
\end{aligned}
$$

■

Thus, using the inapproximability result of [7], we obtain the following.

Theorem 4 *For any $\epsilon > 0$, bidirected MESSCS in symmetric graphs is not approximable within $96/95 - \epsilon$, unless $P = NP$.*

3 Multicasting and broadcasting in symmetric graphs

In this section we present logarithmic approximation algorithms for MEMT and MEBT in symmetric graphs. The algorithms use a reduction of instances of MEMT to instances of NWST.

Consider an instance I_{MEMT} of MEMT which consists of a complete directed graph $G = (V, E)$, a symmetric edge cost function $c : E \to R^+$, a root node $v_0 \in V$ and a set of terminals $D \subseteq V - \{v_0\}$.

We construct an instance I_{NWST} of NWST, which consists of an undirected graph $H = (U, A)$, a node weight function $c' : U \to R^+$ and a set of terminals $D' \subseteq U$. For a node $v \in V$, we denote by n_v the number of different edge costs in the edges directed out of v, and, for $i = 1, ..., n_v$, we denote by $X_i(v)$ the i–th smallest edge cost among the edges directed out of v. The set of nodes U consists of n disjoint sets of nodes called *supernodes*. Each supernode corresponds to a node of V. The supernode

Z_v corresponding to node $v \in V$ has the following $n_v + 1$ nodes: an *input node* $Z_{v,0}$ and n_v *output nodes* $Z_{v,1}, ..., Z_{v,n_v}$. For each pair of nodes $u, v \in V$, the set of edges A contains an edge between the output node $Z_{u,i}$ and the input node $Z_{v,0}$ such that $X_i(u) \geq c(u, v)$. Also, for each node $v \in V$, A contains an edge between the input node $Z_{v,0}$ and each output node $Z_{v,i}$, for $i = 1, ..., n_v$. The cost function c' is defined as $c'(Z_{v,0}) = 0$ for the input nodes and as $c'(Z_{v,i}) = X_i(v)$ for $i = 1, ..., n_v$, for the output nodes. The set of terminals D' is defined as $D' = \{Z_{v,0} \in U | v \in D \cup \{v_0\}\}$. The reduction is depicted in Figure 3.

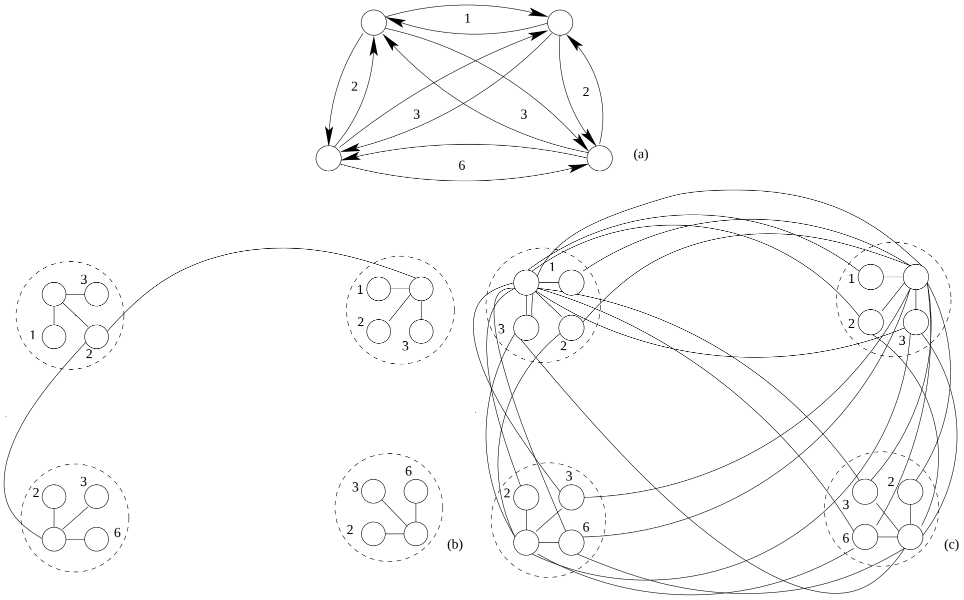

Figure 3: The reduction to Node–Weighted Steiner Tree. (a) The graph G of an instance of MEMT. (b) The graph H of the corresponding instance of NWST. Each large cycle indicates a supernode. Only the edges incident to the node of weight 2 of the upper left supernode are shown. These edges are those which correspond to edges in (a) of cost at most 2, directed out of the left upper node. (c) The graph H of the corresponding instance of NWST.

Consider a subgraph $F = (S, A')$ of H which is a solution for I_{NWST}. We compute a spanning tree $T' = (S, A'')$ of F and, starting from $Z_{v_0,0}$, we compute a Breadth First Search (BFS) numbering of the nodes of T'. For each $v \in S$, we denote by $m(v)$ the BFS number of v. We construct a tree $T = (V, E')$ which, for each edge of F between a node $Z_{u,i}$ of supernode Z_u and a node $Z_{v,j}$ of another supernode Z_v such that $m(Z_{u,i}) < m(Z_{v,j})$, contains a directed edge from u to v. The output of our algorithm is the weight assignment w defined as $w(u) = \max_{(u,v) \in T} c(u, v)$ if u has at least one outgoing edge in T, and $w(u) = 0$, otherwise. We show the following lemma.

Lemma 5 *If F is a ρ–approximate solution to I_{NWST}, then w is a 2ρ–approximate solution to I_{MEMT}.*

Proof: First, we can easily see that w is a solution to MEMT for I_{MEMT}. Since the tree T' is a spanning tree of graph F which is a solution to I_{NWST}, it also maintains the connectivity requirements of I_{NWST}. Thus, by its construction, the tree T maintains the connectivity requirements of I_{MEMT} (i.e., it contains a directed path from node v_0 to each terminal $u \in D$). Clearly, the transmission graph G_w defined by the weight assignment w contains T as a subgraph and, hence, it is a solution for I_{MEMT}.

Now, assume that F is a ρ–approximate solution for I_{NWST}, i.e., $COST(I_{NWST}) \leq \rho \cdot OPT(I_{NWST})$, where $COST(I_{NWST})$ is the cost of the solution F and $OPT(I_{NWST})$ is the cost of the optimal solution of NWST for I_{NWST}. We will show that the cost $COST(I_{MEMT})$ of the solution w is upper-bounded by $2 \cdot COST(I_{NWST})$ and, furthermore, that $OPT(I_{NWST})$ is upper-bounded by the cost $OPT(I_{MEMT})$ of the optimal solution of MEMT for I_{MEMT}. In this way, we will obtain that w is a 2ρ–approximate solution for I_{MEMT} since

$$COST(I_{MEMT}) \leq 2 \cdot COST(I_{NWST}) \leq 2\rho \cdot OPT(I_{NWST}) \leq 2\rho \cdot OPT(I_{MEMT}).$$

Observe that, for each directed edge between u and v in T, there exists an edge between a node of the supernode Z_u and a node of the supernode Z_v in F. By the definition of the graph H and of the cost function c', at least one of the nodes of Z_u and Z_v which belong in S has weight at least $c(u,v)$. In other words, for each edge between u and v in T, it is

$$c(u,v) \leq \sum_{v' \in Z_u \cap S} c'(v') + \sum_{v' \in Z_v \cap S} c'(v').$$

Consider the set of edges B which, for each node u of T which is not a leaf, contains the edge (u,v) having the largest cost among the costs of the edges of T going out of u. Observe that each node of V is adjacent to at most two edges of B. We can express the cost of the solution w as

$$\begin{aligned}
COST(I_{MEMT}) &= \sum_{u \in V} w(u) \\
&= \sum_{u \in V'} \max_{v:(u,v) \in E'} \{c(u,v)\} \\
&= \sum_{(u,v) \in B} c(u,v) \\
&\leq \sum_{(u,v) \in B} \left(\sum_{v' \in Z_u \cap S} c'(v') + \sum_{v' \in Z_v \cap S} c'(v') \right) \\
&\leq 2 \sum_{v' \in S} c'(v') \\
&= 2 \cdot COST(I_{NWST}),
\end{aligned}$$

where V' denotes the set of nodes of V which have at least one outgoing edge in T.

Next, in order to show that $OPT(I_{NWST})$ is upper-bounded by $OPT(I_{MEMT})$, we will construct a solution for I_{NWST} of cost $OPT(I_{MEMT})$. Let w' be an optimal weight assignment for I_{MEMT}, and consider the transmission graph $G_{w'}$. Since w' is optimal, it is $w'(u) = \max_{v \in V:(u,v) \in E'}\{c(u,v)\}$ if u has at least one outgoing edge in $G_{w'}$, and $w'(u) = 0$, otherwise. For each node u having at least one outgoing edge in $G_{w'}$, we define $\chi(u)$ to be such that $c'(Z_{u,\chi(u)}) = w'(u)$. We construct the subgraph $F' = (S', A')$ of H as follows. The set of nodes S' contains all input nodes and the output nodes $Z_{u,\chi(u)}$ for each node u of V which has at least one outgoing edge in $G_{w'}$. The set of edges A' contains an edge between the output node $Z_{u,\chi(u)}$ and the input node $Z_{v,0}$ for each directed edge (u,v) of $G_{w'}$ (this edge does exist since $c(u,v) \leq w'(u) = c'(Z_{u,\chi(u)})$), and an edge between the input node $Z_{u,0}$ and the output node $Z_{u,\chi(u)}$ for each node u having at least one outgoing edge in $G_{w'}$.

Observe that for each directed path $v_0, v_1, v_2, ..., v_t, u$ from the node v_0 to a terminal u of D in $G_{w'}$, the graph F' contains the path $Z_{v_0,0}$, $Z_{v_0,\chi(v_0)}$, $Z_{v_1,0}$, $Z_{v_1,\chi(v_1)}$, $Z_{v_2,0}$, $Z_{v_2,\chi(v_2)}$, ..., $Z_{v_t,0}$, $Z_{v_t,\chi(v_t)}$, $Z_{u,0}$ between nodes $Z_{v_0,0}$ and $Z_{u,0}$. Thus, each pair of terminals of D' is connected through a path in F'. Hence, the graph F' is indeed a solution to NWST, for instance I_{NWST}. Its cost is

$$\sum_{u \in S'} c'(u) = \sum_{Z_{u,\chi(u)}:u \in V'} c'(Z_{u,\chi(u)})$$

$$= \sum_{u \in V} w'(u)$$
$$= OPT(I_{MEMT}),$$

where V' denotes the set of nodes of V which have at least one outgoing edge in $G_{w'}$. ∎

In [12], Guha and Khuller present a $1.35 \ln k$–approximation algorithm for NWST, where k is the number of terminals in the instance of NWST. Given an instance I_{MEMT} of MEMT with a set of terminals D, the corresponding instance I_{NWST} has $|D|+1$ terminals. Thus, the cost of the solution of I_{MEMT} is within $2 \cdot 1.35 \ln(|D|+1) = 2.7 \ln(|D|+1)$ of the optimal solution. The next theorem summarizes the discussion of this section. We remind that MEBT is the special case of MEMT with $D = V - \{v_0\}$.

Theorem 6 *There exist a $2.7 \ln(|D|+1)$– and a $2.7 \ln n$–approximation algorithm for MEMT and MEBT in symmetric graphs, respectively.*

4 Approximating MEIMT and MEIBT

In this section, we show that MEIMT is equivalent to DST. As corollaries, we obtain approximability and inapproximability results for MEIMT in symmetric and asymmetric graphs and optimal solution of MEIBT in polynomial time.

Assume that we have an instance I_{MEIMT} of MEIMT defined by a complete directed graph $G = (V, E)$, an edge cost function $c : E \to R^+$, a root node $v_0 \in V$ and a set of terminals $D \subseteq V - \{v_0\}$. Consider the instance I_{DST} of DST that consists of G, the edge cost function $c' : E \to R^+$ defined as $c'(u, v) = c(v, u)$ for any edge $(u, v) \in E$, the set of terminals D and the root node v_0. Also, we may start by an instance I_{DST} of DST and construct I_{MEIMT} in the same way. We can prove the following lemma.

Lemma 7 *A ρ-approximate solution for I_{DST} reduces in polynomial time to a ρ-approximate solution for I_{MEIMT} and a ρ-approximate solution for I_{MEIMT} reduces in polynomial time to a ρ-approximate solution for I_{DST}.*

Proof: We will show that, starting from a solution for one problem, we can construct a solution for the other of at most the same cost. Note that this implies that the optimal solutions of both I_{DST} and I_{MEIMT} have the same cost.

Consider a solution of I_{MEIMT}, i.e., a weight assignment w to the nodes of V such that the transmission graph G_w defined by this weight assignment satisfies the connectivity requirements of I_{MEIMT}. Due to these connectivity requirements, the transmission graph G_w contains a directed tree T spanning the nodes of D and directed towards u_0. The cost of this solution is at least

$$\sum_{u \in V} \max_{v:(u,v) \in T} \{c(u, v)\} = \sum_{(u,v) \in T} c(u, v),$$

since each node of T other than u_0 has out–degree one. Now the tree T' which contains the opposite directed edge for each edge of T is clearly a solution for I_{DST} of cost

$$\sum_{(v,u) \in T'} c'(v, u) = \sum_{(u,v) \in T} c(u, v).$$

Similarly, starting from a solution T' for I_{DST} of cost $\sum_{(v,u) \in T'} c'(v, u)$, we construct a directed tree T by taking the opposite directed edges for each directed edge of T'. This tree maintains the connectivity requirements of I_{MEIMT}. Thus, by setting $w(u) = c(u, v)$ for each $u \in T$ but the root node u_0, we obtain a weight assignment w such that the transmission graph G_w contains T as a subgraph, and, thus, is a solution for I_{MEIMT} of cost

$$\sum_{u \in V} w(u) = \sum_{(u,v) \in T} c(u, v) = \sum_{(v,u) \in T'} c'(v, u).$$

■

As corollaries, using the approximability and inapproximability results of [6] and [14], we obtain that MEIMT is approximable within $O(|D|^{\epsilon})$ and inapproximable within $O(\ln^{2-\epsilon} n)$, for any constant $\epsilon > 0$. Notice that DST in symmetric graphs is equivalent to ST. Thus, using the approximability and inapproximability results of [19] and [7], we obtain that MEIMT in symmetric graphs is approximable within 1.55 and inapproximable within $96/95 - \epsilon$ for any $\epsilon > 0$. Also, instances of DST having all non-root nodes as terminals are actually instances of Minimum Spanning Arborescence which is known to be computable in polynomial time [10]. Thus, MEIBT can be solved in polynomial time (even in asymmetric graphs).

5 Approximating MEMT and MESSCS

In this section, we first show that MEMT and MESSCS are as hard to approximate as DST. Then, we present a method for approximating MESSCS.

Consider an instance I_{DST} of DST that consists of a directed graph $G = (V, E)$ with an edge cost function $c : E \to R^+$, a root node v_0 and a set of terminals $D \subseteq V - \{v_0\}$. Without loss of generality, we may assume that G is a complete directed graph with some of its edges having infinite cost.

We construct the instance I_{MEMT} of MEMT which consists of a complete directed graph $H = (U, A)$ with edge cost function $c' : A \to R^+$, a root node $v'_0 \in U$ and a set of terminals $D' \subseteq U - \{v'_0\}$. The set of nodes U has a node h_v for each node $v \in V$ and a node $h_{(u,v)}$ for each directed edge (u, v) of E. For each directed edge (u, v) of E, the directed edge $(h_u, h_{(u,v)})$ of A has zero cost and the directed edge $(h_{(u,v)}, h_v)$ of A has cost $c'(h_{(u,v)}, h_v) = c(u, v)$, while all other edges of A have infinite cost. This construction is presented in Figure 4. The set of terminals is defined as $D' = \{h_u \in U | u \in D\}$, while $v'_0 = h_{v_0}$. We show the following lemma.

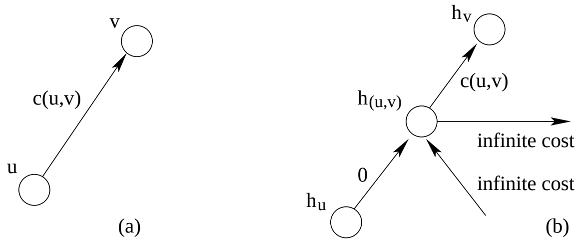

Figure 4: An edge in I_{DST} (a) and the corresponding structure in I_{MEMT} (b). In (b), edges directed out of $h_{(u,v)}$ which are not incident to h_v, as well as edges which are not incident to h_u and are destined for $h_{(u,v)}$ have infinite cost.

Lemma 8 *A ρ–approximate solution to I_{MEMT} reduces in polynomial time to a ρ–approximate solution to I_{DST}.*

Proof: Consider a solution w to I_{MEMT} of cost $COST(I_{MEMT})$. Let $H_w = (U, E(H_w))$ be the corresponding transmission graph. We construct a solution $F = (V, E')$ to I_{DST} with cost not greater than $COST(I_{MEMT})$ as follows. A directed edge (u, v) is contained in E' only if the edge $(h_{(u,v)}, h_v)$ belongs to the transmission graph H_w.

Observe that for each node u of V, the transmission graph contains all edges directed out of h_u having zero cost. Thus, for each directed path from h_{u_0} to a node h_v in the transmission graph, F contains a directed path from u_0 to v. Thus, F is a feasible solution of I_{DST}.

Now, assume that w is a ρ–approximate solution for I_{MEMT}, i.e., $COST(I_{MEMT}) \leq \rho \cdot OPT(I_{MEMT})$, where $COST(I_{MEMT})$ is the cost of the solution w and $OPT(I_{MEMT})$ is the cost of the optimal solution of MEMT for I_{MEMT}. We will show that $COST(I_{DST})$ is upper-bounded by $COST(I_{MEMT})$, where $COST(I_{DST})$ denotes the cost of F and, furthermore, that $OPT(I_{MEMT})$ is upper-bounded by $OPT(I_{DST})$, where $OPT(I_{DST})$ denotes the cost of the optimal solution of DST for I_{DST}. In this way, we will obtain that F is a ρ–approximate solution for I_{DST} since

$$COST(I_{DST}) \leq COST(I_{MEMT}) \leq \rho \cdot OPT(I_{MEMT}) \leq \rho \cdot OPT(I_{DST}).$$

Indeed, the cost of the solution F is

$$\begin{aligned} COST(I_{DST}) &= \sum_{(u,v)\in E'} c(u,v) \\ &= \sum_{(h_{(u,v)},h_v)\in E(H_w)} c'(h_{(u,v)}, h_v) \\ &\leq \sum_{y\in U} w(y) \\ &= COST(I_{MEMT}) \end{aligned}$$

Next, we show that $OPT(I_{MEMT})$ is upper-bounded by $OPT(I_{DST})$. Consider an optimal solution $F' = (V, E'')$ for I_{DST}. We construct a solution w' for I_{MEMT} as follows. For each directed edge $(u, v) \in E''$, we assign to node $h_{(u,v)}$ weight $w'(h_{(u,v)}) = c'(h_{(u,v)}, h_v) = c(u, v)$, while all other nodes are assigned zero weight. Clearly, w' is a feasible solution, since for each edge $(u, v) \in E''$, there exists a directed path from h_u to h_v in the transmission graph $H_{w'}$. Thus, the cost of the optimal solution of I_{MEMT} is

$$\begin{aligned} OPT(I_{MEMT}) &\leq \sum_{y\in U} w'(y) \\ &= \sum_{(u,v)\in E''} w'(h_{(u,v)}) \\ &= \sum_{(u,v)\in E''} c(u,v) \\ &= OPT(I_{DST}) \end{aligned}$$

■

We construct the instance I_{MESSCS} of MESSCS which consists of the graph G, the set of terminals $D \cup v_0$, and an edge cost function $c'' : E \rightarrow R^+$ defined as follows. For each directed edge (u, v) of E such that $u \neq v_0$, it is $c''(u, v) = c(v, u)$, while all edges of E directed out of v_0 have zero cost. An example of this construction is presented in Figure 5. We can show the following.

Lemma 9 *A ρ–approximate solution to I_{MESSCS} reduces in polynomial time to a ρ–approximate solution to I_{DST}.*

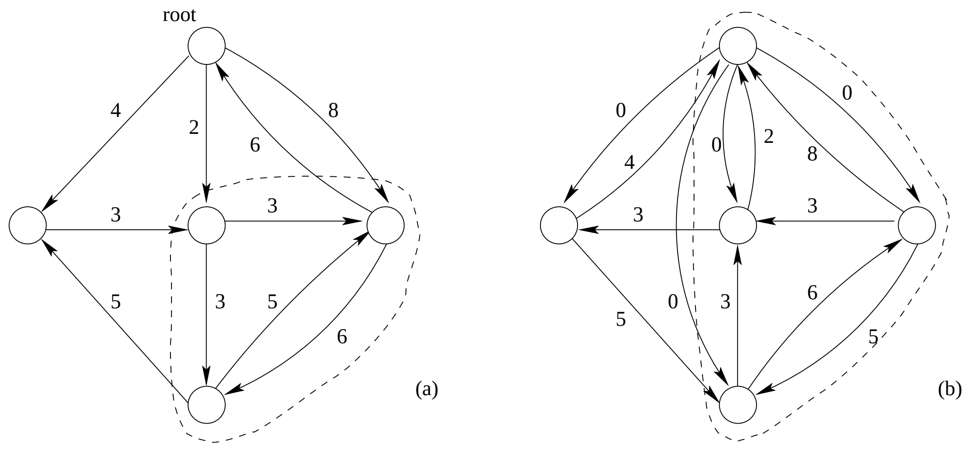

Figure 5: Transforming an instance of I_{DST} (a) to an instance of I_{MESSCS} (b). Dashed closed lines indicate the sets of terminals.

Proof: Consider a solution w to I_{MESSCS} of cost $COST(I_{MESSCS})$. Let $G_w = (V, E(G_w))$ be the corresponding transmission graph. We will construct a solution $F = (V, E')$ to I_{DST} with cost not greater than $COST(I_{MESSCS})$. For each edge (u, v) in the transmission graph G_w, except those edges that are directed out of u_0, we add to E' the opposite directed edge (v, u). Observe that F is a feasible solution to I_{DST}, since for each path from a node $v \in V - \{u_0\}$ to node u_0 in G_w, there exists a path from u_0 to v in E'.

Now, assume that w is a ρ–approximate solution for I_{MESSCS}, i.e., $COST(I_{MESSCS}) \leq \rho \cdot OPT(I_{MESSCS})$, where $COST(I_{MESSCS})$ is the cost of the solution w and $OPT(I_{MESSCS})$ is the cost of the optimal solution of MESSCS for I_{MESSCS}. We will show that $COST(I_{DST})$ is upper-bounded by $COST(I_{MESSCS})$, where $COST(I_{DST})$ denotes the cost of F and, furthermore, that $OPT(I_{MESSCS})$ is upper-bounded by $OPT(I_{DST})$, where $OPT(I_{DST})$ denotes the cost of the optimal solution of DST for I_{DST}. In this way, we will obtain that F is a ρ–approximate solution for I_{DST} since

$$COST(I_{DST}) \leq COST(I_{MESSCS}) \leq \rho \cdot OPT(I_{MESSCS}) \leq \rho \cdot OPT(I_{DST}).$$

Indeed, the cost of the solution F is

$$\begin{aligned}
COST(I_{DST}) &= \sum_{(u,v)\in E'} c(u,v) \\
&\leq \sum_{(v,u)\in E(G_w)} c''(v,u) \\
&\leq \sum_{u\in V} w(u) \\
&= COST(I_{MESSCS})
\end{aligned}$$

Next, we show that $OPT(I_{MESSCS})$ is upper-bounded by $OPT(I_{DST})$. Consider an optimal solution $F' = (V, E'')$ for I_{DST}. We will construct a solution w' for I_{MESSCS} with cost not greater than $OPT(I_{DST})$.

For each directed edge $(u, v) \in E''$, we assign to node v weight $w'(v) = c''(v, u) = c(u, v)$, while all other nodes are assigned zero weight. For every edge $(u, v) \in E''$, the transmission graph $G_{w'}$ contains its opposite directed edge (v, u), thus ensuring that every node in $G_{w'}$ can reach the root node u_0. Also, since each edge directed out of u_0 has zero cost in I_{MESSCS}, node u_0 can reach all other nodes in the transmission graph, therefore w' is a feasible solution. Thus, the cost of the optimal solution for I_{MESSCS} is

$$\begin{aligned} OPT(I_{MESSCS}) &\leq \sum_{u \in V} w'(u) \\ &= \sum_{(v,u) \in E''} c(v,u) \\ &= OPT(I_{DST}), \end{aligned}$$

where the first equality holds since each node of V has in–degree at most one in the optimal solution F'. ■

Using the inapproximability result for DST [14], we obtain the following.

Theorem 10 *For any $\epsilon > 0$, MEMT and MESSCS are not approximable within $O(\ln^{2-\epsilon} n)$, unless $NP \subseteq ZTIME(n^{polylog(n)})$.*

We now present a method for approximating MESSCS. Let I_{MESSCS} be an instance of I_{MESSCS} that consists of a complete directed graph $G = (V, E)$ with edge cost function $c : E \to R^+$ and a set of terminals $D \subseteq V$. Pick an arbitrary node $v_0 \in D$ and let I_{MEMT} and I_{MEIMT} be the instances of MEMT and MEIMT, respectively, consisting of the graph G with edge cost function c, the root node v_0 and the set of terminals $D - \{v_0\}$.

Assume that we have weight assignments w_1 and w_2 to the nodes of V which are solutions for I_{MEMT} and I_{MEIMT}, respectively. Construct the weight assignment w_3 defined as $w_3(u) = \max\{w_1(u), w_2(u)\}$ for every $u \in V$. An example of this construction is presented in Figure 6. We show the following lemma.

Lemma 11 *If the weight assignments w_1 and w_2 are ρ_1- and ρ_2- approximate solutions for I_{MEMT} and I_{MEIMT} respectively, then the weight assignment w_3 is a $(\rho_1 + \rho_2)$–approximate solution to I_{MESSCS}.*

Proof: First observe that the transmission graph G_{w_3} maintains the connectivity requirements of I_{MEMT} and I_{MEIMT}, and, hence, it maintains the connectivity requirements of I_{MESSCS}. Thus, w_3 is a solution to I_{MESSCS}.

We denote by $OPT(I_{MEMT})$, $OPT(I_{MEIMT})$, and $OPT(I_{MESSCS})$ the cost of the optimal solutions to I_{MEMT}, I_{MEIMT}, and I_{MESSCS}, respectively. We also denote by $COST(I_{MEMT})$ and $COST(I_{MEIMT})$ the cost of the solutions w_1 and w_2. Then the cost of w_3 is

$$\begin{aligned} COST(I_{MESSCS}) &= \sum_{u \in V} w_3(u) \\ &= \sum_{u \in V} \max\{w_1(u), w_2(u)\} \\ &\leq \sum_{u \in V} w_1(u) + \sum_{u \in V} w_2(u) \\ &= COST(I_{MEMT}) + COST(I_{MEIMT}) \\ &\leq \rho_1 \cdot OPT(I_{MEMT}) + \rho_2 \cdot OPT(I_{MEIMT}) \\ &\leq (\rho_1 + \rho_2) \cdot \max\{OPT(I_{MEMT}), OPT(I_{MEIMT})\} \\ &\leq (\rho_1 + \rho_2) \cdot OPT(I_{MESSCS}) \end{aligned}$$

where the last inequality holds since the connectivity requirements of I_{MEMT} and I_{MEIMT} are subsets of the connectivity requirements of I_{MESSCS}. ■

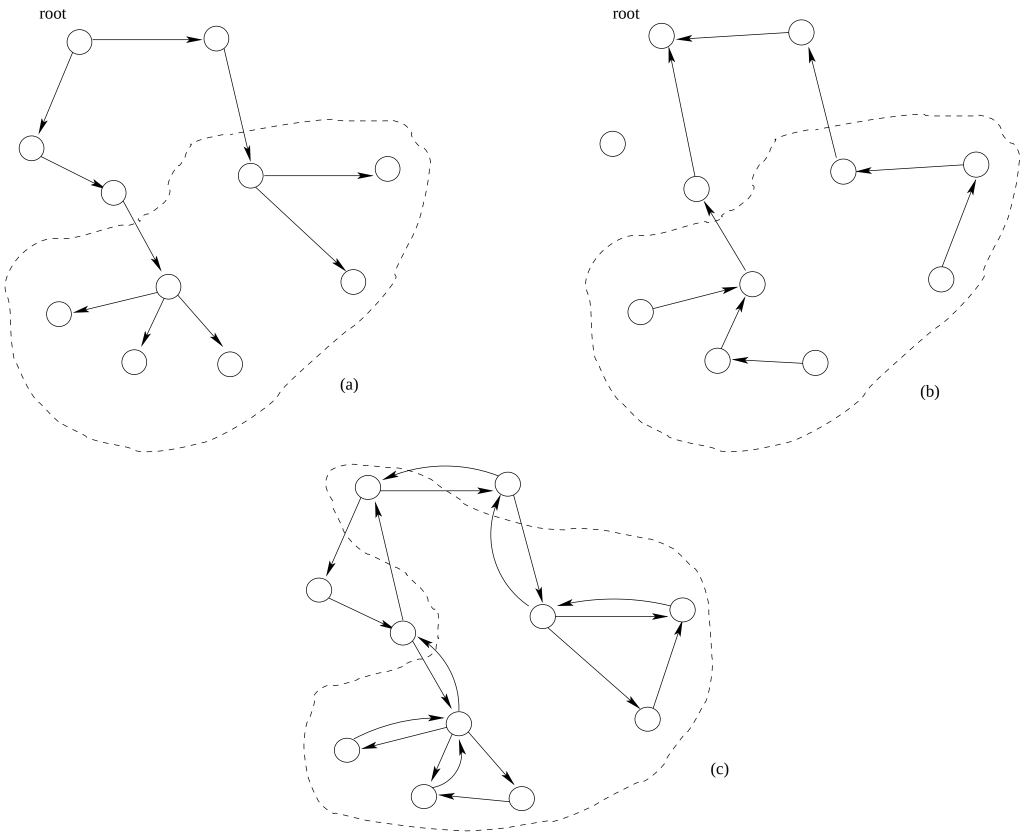

Figure 6: An example of combining solutions for I_{MEMT} (a) and I_{MEIMT} (b) in order to construct a solution for I_{MESSCS} (c). Dashed closed lines indicate the sets of terminals.

We can solve I_{MEMT} and I_{MEIMT} using the $O(|D|^\epsilon)$-approximation algorithm of Liang [17] and the $O(|D|^\epsilon)$-approximation algorithm of [6] for DST. In this way we obtain the following.

Theorem 12 *For any $\epsilon > 0$, there exists an $O(|D|^\epsilon)$-approximation algorithm for MESSCS.*

Note that the algorithm of Liang for approximating MEMT actually computes a solution to an instance of DST with $O(n^2)$ nodes. This means that a polylogarithmic approximation algorithm for DST would immediately yield polylogarithmic approximation algorithms for MEMT and MESSCS.

6 Logarithmic approximations for MEBT and MESCS

In this section we show that MEBT and MESCS can be approximated within a logarithmic factor. These results are optimal within constant factors.

Liang in [17] presents an intuitive reduction for transforming an instance I_{MEBT} of MEBT into an instance I_{DST} of DST in such a way that a ρ–approximate solution for I_{DST} implies a ρ–approximate solution for I_{MEBT}.

We describe this reduction here. Assume that I_{MEBT} consists of a complete directed graph $G = (V, E)$ with an edge cost function $c : E \to R^+$ and a root node $r \in V$. Then, the instance I_{DST} consists of a directed graph $H = (U, A)$ with an edge cost function $c' : A \to R^+$, a root node $r' \in U$ and a set of terminals $D \subseteq U - \{r'\}$. For a node $v \in V$, we denote by n_v the number of different edge costs in the edges directed out of v, and, for $i = 1, ..., n_v$, we denote by $X_i(v)$ the i–th smallest edge cost among the edges directed out of v. For each node $v \in V$, the set of nodes U contains $n_v + 1$ nodes $Z_{v,0}, Z_{v,1}, ..., Z_{v,n_v}$. For each directed edge $(v, u) \in E$ and for $i = 1, ..., n_v$, the set of edges A contains a directed edge of zero cost from $Z_{v,i}$ to $Z_{u,0}$ if $X_i(v) \geq c(v, u)$. Also, for each node $v \in V$, and $i = 1, ..., n_v$, the set of edges A contains a directed edge from $Z_{v,0}$ to $Z_{v,i}$ of cost $c'(Z_{v,0}, Z_{v,i}) = X_i(v)$. An example of this construction is presented in Figure 7. The set of terminals is defined by $D = \{Z_{v,0} | v \in V - \{r\}\}$ and $r' = Z_{r,0}$.

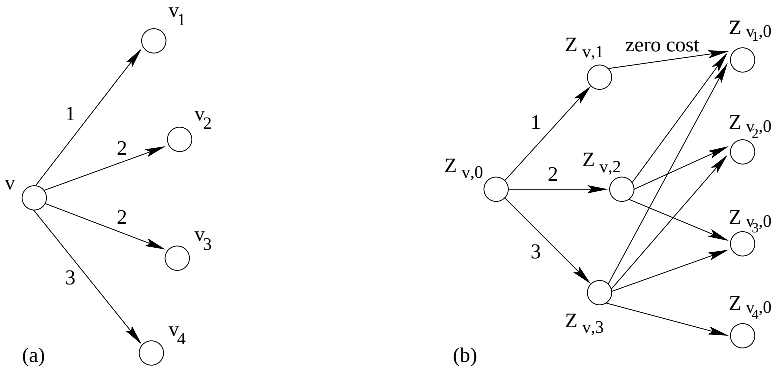

Figure 7: Liang's reduction of MEBT to DST. A node v and its outgoing edges in I_{MEBT} (a) and the corresponding structure in I_{DST} (b). All edges in (b) directed out of $Z_{v,1}$, $Z_{v,2}$, and $Z_{v,3}$ have zero cost.

We use an algorithm proposed by Zosin and Khuller [22] to approximate I_{DST} by repeatedly solving instances of the Minimum Density Directed Tree (MDDT) problem. An instance of MDDT is defined in the same way as instances of DST and the objective is to compute a tree directed out of the root node such that the ratio of the cost of the tree over the number of terminals it spans is minimized. The algorithm of [22] repeatedly solves instances I^i_{MDDT} of MDDT derived by the instance I_{DST}. The instance I^1_{MDDT} is defined by the graph H with edge cost function c, the set of terminals $D_1 = D$ and the root node $r_1 = r'$. Initially, the algorithm sets $i = 1$. While $D_i \neq \emptyset$, it repeats the following. It finds a solution T to I^i_{MDDT} that consists of a tree $T_i = (V(T_i), E(T_i))$, defines the instance I^{i+1}_{MDDT} by contracting the nodes of T_i into the root node r_{i+1} and by setting $D_{i+1} = D_i \backslash V(T_i)$, and increments i by 1.

Zosin and Khuller [22] show that if the solution T_i is a ρ–approximate solution for I^i_{MDDT} in each iteration i, then the union of the trees T_i computed in all iterations is an $O(\rho \ln n)$–approximate solution for I_{DST}. They also show how to find a $(d+1)$–approximate solution for I^i_{MDDT} if the graph obtained when removing the terminals from G has depth d. Observe that, given an instance I_{MEBT} of MEBT, the graph H obtained by applying the reduction of Liang is bipartite, since there is no edge between nodes of $D \cup \{r'\}$ and between nodes of $V - (D \cup \{r'\})$. Thus, the graph obtained by removing the terminals of D from H has depth 1. Following the reasoning of [22], we obtain the following result.

Theorem 13 *There exists an $O(\ln n)$-approximation algorithm for MEBT.*

Now, following a similar technique with the one we used in the previous section for approximating MESSCS, we can solve any instance of MESCS by solving an instance of MEBT (using the $O(\ln n)$-approximation algorithm described above) and an instance of MEIBT (this can be done optimally in polynomial time as we described in Section 4), and then merging the two solutions. In this way we obtain the following result.

Theorem 14 *There exists an $O(\ln n)$-approximation algorithm for MESCS.*

7 Approximating bidirected MESS and related problems

In the following, we present a logarithmic approximation algorithm for bidirected MESS. The algorithm uses a reduction of instances of bidirected MESS to instances of NWSF. The main idea behind this reduction is similar to the one used in Section 3. However, both the constructions and the analysis have subtle differences.

Consider an instance I_{bMESS} of bidirected MESS which consists of a complete directed graph $G = (V, E)$, an edge cost function $c : E \rightarrow R^+$ and a set of terminals $D \subseteq V$ partitioned into p disjoint subsets $D_1, D_2, ..., D_p$.

We construct an instance I_{NWSF} of NWSF consisting of an undirected graph $H = (U, A)$, a node weight function $c' : U \rightarrow R^+$ and a set of terminals $D' \subseteq U$ partitioned into p disjoint sets $D'_1, D'_2, ..., D'_p$. For a node $v \in V$, we denote by n_v the number of different edge costs in the edges directed out of v, and, for $i = 1, ..., n_v$, we denote by $X_i(v)$ the i–th smallest edge cost among the edges directed out of v. The set of nodes U consists of n disjoint sets of nodes called *supernodes*. Each supernode corresponds to a node of V. The supernode Z_v corresponding to node $v \in V$ has the following $n_v + 1$ nodes: a *hub node* $Z_{v,0}$ and n_v *bridge nodes* $Z_{v,1}, ..., Z_{v,n_v}$. For each pair of nodes $u, v \in V$, the set of edges A contains an edge between the bridge nodes $Z_{v,i}$ and $Z_{u,j}$ such that $X_i(u) \geq c(u, v)$ and $X_j(v) \geq c(v, u)$. Also, for each node $v \in V$, A contains an edge between the hub node $Z_{v,0}$ and each bridge node $Z_{v,i}$, for $i = 1, ..., n_v$. The cost function c' is defined as $c'(Z_{v,0}) = 0$ for the hub nodes and as $c'(Z_{v,i}) = X_i(v)$ for $i = 1, ..., n_v$, for the bridge nodes. The set of terminals D' is defined as $D' = \cup_i D'_i$ where $D'_i = \{Z_{v,0} \in U | v \in D_i\}$. The reduction is depicted in Figure 8.

Consider a subgraph $F = (S, A')$ of H which is a solution for I_{NWSF}. We construct a weight assignment w on the nodes of G by setting $w(v) = 0$ if S contains no node from supernode Z_v, and $w(v) = \max_{u \in (Z_v \cap S)} c'(u)$, otherwise. We show the following lemma.

Lemma 15 *If F is a ρ–approximate solution to I_{NWSF}, then w is a ρ–approximate solution to I_{bMESS}.*

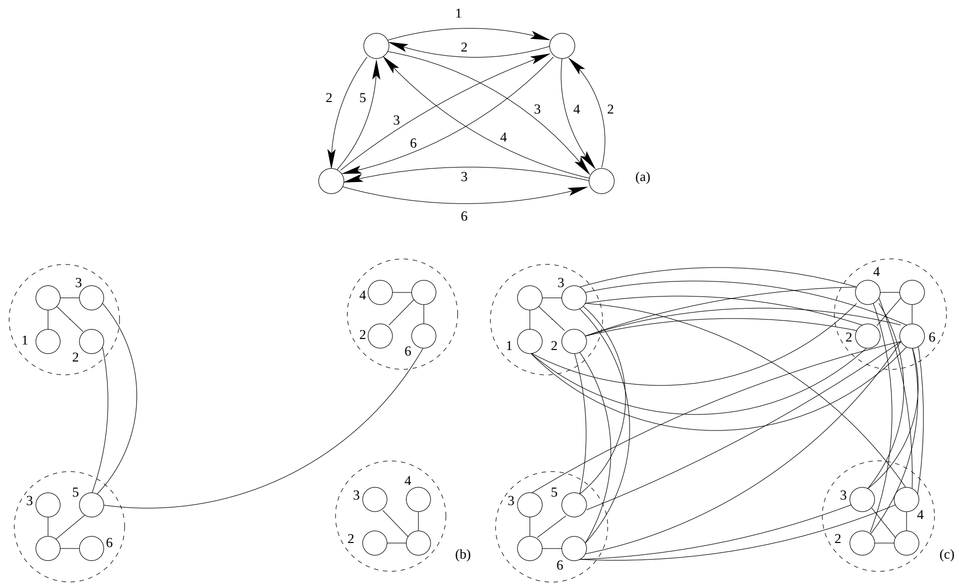

Figure 8: The reduction to Node–Weighted Steiner Forest. (a) The graph G of an instance of bMESS. (b) The graph H of the corresponding instance of NWSF. Each large cycle indicates a supernode. Only the edges incident to the node of weight 5 of the lower left supernode are shown. (c) The graph H of the corresponding instance of NWSF.

Proof: Consider the bidirected graph $G' = (V, E')$ where the set of edges E' contains the opposite directed edges (u, v) and (v, u) if A' contains an edge between a bridge node $Z_{u,i}$ of supernode Z_u and a bridge node $Z_{v,j}$ of supernode Z_v. Since, by construction, it is $w(u) \geq c'(Z_{u,i}) \geq c(u, v)$ and $w(v) \geq c'(Z_{v,j}) \geq c(v, u)$, the transmission graph G_w contains G' as a subgraph.

Now consider a path connecting the hub nodes $Z_{v,0}$ and $Z_{u,0}$. Following the path from node $Z_{v,0}$ to node $Z_{u,0}$, assume that it contains nodes of supernodes $Z_v, Z_{v_1}, ..., Z_{v_t}, Z_u$. The path enters and leaves a supernode through edges between bridge nodes. Hence, we may construct the directed path $v, v_1, ..., v_t, u$ in G'. Since G' is bidirected, it also contains the path $u, v_t, ..., v_1, v$.

Since F contains a path between hub nodes $Z_{v_i,0}$ and $Z_{v_j,0}$ if $Z_{v_i,0}, Z_{v_j,0} \in D'_k$ for some k, this means that G' contains a directed path from v_i to v_j and a directed path from v_j to v_i for each pair of nodes v_i, v_j belonging to the same set D_k, for some k. Thus, the transmission graph G_w contains the bidirected graph G' as a subgraph which maintains the connectivity requirements for I_{bMESS} and, hence, w is a solution of bidirected MESS for I_{bMESS}.

Now, assume that F is a ρ–approximate solution for I_{NWSF}, i.e., $COST(I_{NWSF}) \leq \rho \cdot OPT(I_{NWSF})$, where $COST(I_{NWSF})$ is the cost of the solution F and $OPT(I_{NWSF})$ is the cost of the optimal solution of NWSF for I_{NWSF}. We will show that $COST(I_{bMESS})$ is upper-bounded by $COST(I_{NWSF})$, where $COST(I_{bMESS})$ denotes the cost of w and, furthermore, that $OPT(I_{NWSF})$ is upper-bounded by $OPT(I_{bMESS})$ denoting the cost of the optimal solution of bidirected MESS for I_{bMESS}. In this way, we will obtain that w is a ρ–approximate solution for I_{bMESS} since

$$COST(I_{bMESS}) \leq COST(I_{NWSF}) \leq \rho \cdot OPT(I_{NWSF}) \leq \rho \cdot OPT(I_{bMESS}).$$

Indeed, the cost of the solution w is

$$\begin{aligned} COST(I_{bMESS}) &= \sum_{v \in V} w(v) \\ &= \sum_{v \in V : Z_v \cap S \neq \emptyset} \max_{u \in Z_v \cap S} \{c'(u)\} \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{v \in V: Z_v \cap S \neq \emptyset} \sum_{u \in Z_v \cap S} c'(u) \\
&= \sum_{v \in S} c'(u) \\
&= COST(I_{NWSF}).
\end{aligned}
$$

Next, we show that $OPT(I_{NWSF})$ is upper-bounded by $OPT(I_{bMESS})$. In order to do this, given an optimal weight assignment w' for I_{bMESS}, we will construct a solution for I_{NWSF} of cost $OPT(I_{bMESS})$.

Consider the transmission graph $G_{w'}$. Since w' is a solution for I_{bMESS}, $G_{w'}$ contains a bidirected graph $G' = (V, E')$ which maintains the connectivity requirements of I_{bMESS}. Since w' is optimal, it is $w'(u) = \max_{v \in V:(u,v) \in E'}\{c(u, v)\}$ if u is connected to at least one node in G', and $w'(u) = 0$, otherwise. For each node u connected to some other node in G', we define $\chi(u)$ to be such that $c'(Z_{u,\chi(u)}) = w'(u)$. We construct the subgraph $F' = (S', A')$ of H as follows. The set of nodes S' contains all terminals of D' and the bridge nodes $Z_{u,\chi(u)}$ for each node u of V which is connected to at least one node in G'. The set of edges A' contains an edge between the bridge nodes $Z_{u,\chi(u)}$ and $Z_{v,\chi(v)}$ for each pair of opposite directed edges between nodes u and v in G' (these edges do exist since $c(u, v) \leq w'(u) = c'(Z_{u,\chi(u)})$ and $c(v, u) \leq w'(v) = c'(Z_{v,\chi(v)})$), and an edge between the hub node $Z_{u,0}$ and the bridge node $Z_{u,\chi(u)}$ (note that this node is well-defined since all terminals are connected to at least one node in G') for each terminal u of D.

We observe that, for any pair of opposite directed paths $u, u_1, u_2, ..., u_t, v$ in G', the graph F' contains a path from $Z_{u,0}$ to $Z_{v,0}$ which traverses the bridge nodes $Z_{u,\chi(u)}$, $Z_{u_1,\chi(u_1)}$, $Z_{u_2,\chi(u_2)}$, ..., $Z_{u_t,\chi(u_t)}$, $Z_{v,\chi(v)}$. Since G' contains a pair of opposite directed paths between nodes v_i and v_j if $v_i, v_j \in D_k$ for some k, this means that F' contains a path between the hub nodes $Z_{v_i,0}$ and $Z_{v_j,0}$ for each pair of hub nodes $Z_{v_i,0}$ and $Z_{v_j,0}$ belonging to the same set D'_k, for some k. Thus, the graph F' is indeed a solution of NWSF for instance I_{NWSF}. Its cost is

$$
\begin{aligned}
\sum_{v \in S'} c'(v) &= \sum_{Z_{u,\chi(u)}: u \in V'} c'(Z_{u,\chi(u)}) \\
&= \sum_{u \in V} w'(u) \\
&= OPT(I_{bMESS}),
\end{aligned}
$$

where V' denotes the set of nodes of V which are connected to at least one node in G'. ∎

In [12], Guha and Khuller present a $1.61 \ln k$–approximation algorithm for NWSF, where k is the number of terminals in the graph. Using this algorithm to solve I_{NWSF}, we obtain a solution of I_{bMESS} which is within $1.61 \ln |D|$ of optimal. Moreover, when $p = 1$ (i.e., when I_{bMESS} is actually an instance of bidirected MESSCS), the instance I_{NWSF} is actually an instance of NWST which can be approximated within $1.35 \ln k$, where k is the number of terminals in the graph [12]. The next theorem summarizes the discussion of this section.

Theorem 16 *There exist an $1.61 \ln |D|$-, an $1.35 \ln |D|$-, and an $1.35 \ln n$-approximation algorithm for bidirected MESS, bidirected MESSCS, and bidirected MESCS, respectively.*

8 Open problems

The wireless network design problems we study in this paper are defined by $0-1$ requirement matrices. A natural extension is to consider matrices with non-negative integer entries r_{ij} denoting that r_{ij}

node–disjoint paths are required from node v_i to node v_j. This extension leads to combinatorial problems that capture important engineering problems related to the design of fault–tolerant wireless networks. Some results in this direction are presented in [13] and [16].

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