The efficiency of fair division*

Ioannis Caragiannis, Christos Kaklamanis, Panagiotis Kanellopoulos, and Maria Kyropoulou

Research Academic Computer Technology Institute and
Department of Computer Engineering and Informatics
University of Patras, 26500 Rio, Greece

Abstract. We study the impact of fairness on the efficiency of allocations. We consider three different notions of fairness, namely proportionality, envy-freeness, and equitability for allocations of divisible and indivisible goods and chores. We present a series of results on the price of fairness under the three different notions that quantify the efficiency loss in fair allocations compared to optimal ones. Most of our bounds are either exact or tight within constant factors. Our study is of an optimistic nature and aims to identify the potential of fairness in allocations.

1 Introduction

Fair division (or fair allocation) dates back to the ancient times and has found applications such as border settlement in international disputes, greenhouse gas emissions reduction, allocation of mineral riches in the ocean bed, inheritance, divorces, etc. In the era of the Internet, it appears regularly in distributed resource allocation and cost sharing in communication networks.

We consider allocation problems in which a set of *goods* or *chores* has to be allocated among several players. Fairness is an apparent desirable property in these situations and means that each player gets a *fair share*. Depending on what the term "fair share" means, different notions of fairness can be defined. An orthogonal issue is *efficiency* that refers to the total happiness of the players. An important notion that captures the minimum efficiency requirement from an allocation is that of *Pareto-efficiency*; an allocation is Pareto-efficient if there is no other allocation that is strictly better for at least one player and is at least as good for all the others.

Model and problem statement. We consider two different allocation scenaria, depending on whether the items to be allocated are goods or chores. In both cases, we distinguish between *divisible* and *indivisible* items.

The problem of allocating divisible goods is better known as *cake-cutting*. In instances of cake-cutting, the term *cake* is used as a synonym of the whole set of goods to be allocated. Each player has a *utility function* on each piece

* This work is partially supported by the European Union under IST FET Integrated Project FP6-015964 AEOLUS and Cost Action IC0602 "Algorithmic Decision Theory", and by a "Caratheodory" basic research grant from the University of Patras.

of the cake corresponding to the happiness of the player if she is allocated the particular piece; this function is non-negative and additive. We assume that the utility of each player for the whole cake is 1. Divisibility means that the cake can be cut in arbitrarily small pieces which can then be allocated to the players. In instances with indivisible goods, the utility function of a player is defined over sets of items; again, utilities are non-negative and additive and the utility of each player for the whole set of items is 1. Each item cannot be cut in pieces and has to be allocated as a whole to some player. Given an allocation, the utility of a player is simply the sum of her utilities over the (pieces of) items she receives. An allocation with n players is proportional if the utility of each player is at least $1/n$. It is envy-free if the utility of a player is not smaller than the utility she would have when exchanging the (pieces of) items she gets with the items of any other player. It is equitable if the utilities of all players are equal. An allocation is *optimal* if it maximizes the total utility of all players, i.e., each (piece of) item is allocated to the player that values it the most (ties are broken arbitrarily).

In instances with divisible chores, each player has a *disutility function* for each piece of the cake which denotes the regret of the player when she is allocated the particular piece. Again, the disutility functions are non-negative and additive and the disutility of a player for the whole cake is 1. The case of indivisible chores is defined accordingly; indivisibility implies that an item cannot be cut into pieces and has to be allocated as a whole to some player. Given an allocation, the disutility of a player is simply the sum of her disutilities over the (pieces of) items she receives. An allocation with n players is proportional if the disutility of each player is at most $1/n$. It is envy-free if the disutility of a player is not larger than the disutility she would have when exchanging the (pieces of) items she gets with the items of any other player. It is equitable if the disutilities of all players are equal. An allocation is optimal if it minimizes the total disutility of all players, i.e., each (piece of) item is allocated to the player that values it the least (ties are broken arbitrarily).

Note that envy-freeness implies proportionality. Furthermore, instances with divisible items always have proportional, envy-free, or equitable allocations. It is not hard to see that this is not always the case for instances with indivisible items. Furthermore, there are instances in which no optimal allocation is fair.

Models similar to ours have been considered in the literature; the focus has been on the design of protocols for achieving proportionality [3, 4, 8], envy-freeness [3, 6, 7], and equitability [3] or on the design of approximation algorithms in settings where fulfilling the fairness objective exactly is impossible [2, 5]. However, the related literature seems to have neglected the issue of efficiency. Although several attempts have been made to characterize fair division protocols in terms of Pareto-efficiency [3], the corresponding results are almost always negative. Most of the existing protocols do not even provide Pareto-efficient solutions and this seems to be due to the limited amount of information they use for the utility functions of the players. Recall that in the case of divisible goods and chores, complete information about the utility or disutility functions of the players may not be compactly representable. Furthermore, Pareto-efficiency is rather unsatisfactory, since it may imply that an allocation is far from optimal.

Instead, in the current paper we are interested in quantifying the decrease of efficiency due to fairness (*price of fairness*). Our study has an optimistic nature and aims to identify the potential of fairness in allocations. We believe that such a study is well-motivated since the knowledge of tight bounds on the price of fairness may detect whether a fair allocation can be improved. In many settings, complete information about the utility functions of the players is available (e.g., in a divorce) and computing an efficient and fair allocation may not be infeasible. Fair allocations can be thought of as counterparts of equilibria in strategic games; hence, our work is similar in spirit to the line of research that studies the price of stability in games [1].

In order to capture the price of fairness, we define the price of proportionality, envy-freeness, and equitability. Given an instance I for the allocation of goods, its price of proportionality (resp., envy-freeness, resp., equitability) is defined as the ratio of the total utility of the players in the optimal allocation for I over the total utility of the players in the best proportional (resp., envy-free, resp., equitable) allocation for I. Similarly, if I is an instance for the allocation of chores, its price of proportionality (resp., envy-freeness, resp., equitability) is defined as the ratio of the total disutility of the players in the best proportional (resp., envy-free, resp., equitable) allocation for I over the total disutility of the players in the optimal allocation for I. The price of proportionality (resp., envy-freeness, resp., equitability) of a class $\mathcal{I}$ of instances is then the maximum price of proportionality (resp., envy-freeness, resp., equitability) over all instances of $\mathcal{I}$. The classes of instances considered in this paper are defined by the number of players, the type of items (goods or chores), and their divisibility property (divisible or indivisible). We remark that, in order for the price of proportionality, envy-freeness, and equitability to be well-defined, in the case of indivisible items, we assume that the class of instances contains only those ones for which proportional, envy-free, and equitable allocations, respectively, do exist.

Overview of results. In this paper we provide upper and lower bounds on the price of proportionality, envy-freeness, and equitability in fair division with divisible and indivisible goods and chores. Our work reveals an almost complete picture. In all subcases except the price of envy-freeness with divisible goods and chores, our bounds are either exact or tight within a small constant factor.

Table 1 summarizes our results. For divisible goods, the price of proportionality is very close to 1 (i.e., $8-4\sqrt{3} \approx 1.072$) for two players and $\Theta(\sqrt{n})$ in general. The price of equitability is slightly worse for two players (i.e., 9/8) and $\Theta(n)$ in general. Our lower bound for the price of proportionality implies the same lower bound for the price of envy-freeness; while a very simple upper bound of $n-1/2$ completes the picture for divisible goods. For indivisible goods, we present an exact bound of $n-1+1/n$ on the price of proportionality while we show that the price of envy-freeness is $\Theta(n)$ in this case. Although our upper bounds follow by very simple arguments, the lower bounds use quite involved constructions. The price of equitability is proven to be finite only for the case of two players. These results are presented in Section 2. For divisible chores, the price of proportionality is 9/8 for two players and $\Theta(n)$ in general while the price of equitability is exactly n. For indivisible chores, we present an exact bound of n on the price of proportionality while both the price of envy-freeness and the price of equitability are infinite. These last results imply that in the case of indivisible chores, envy-freeness and equitability are usually incompatible with efficiency. These results are presented in Section 3. Due to lack of space, many proofs have been omitted.

	LB	UB	$n=2$	LB	UB	$n=2$
Price of	Divisible goods			Indivisible goods		
Proportionality	$\Omega(\sqrt{n})$	$O(\sqrt{n})$	$8-4\sqrt{3}$	$n-1+1/n$	$n-1+1/n$	$3/2$
Envy-freeness	$\Omega(\sqrt{n})$	$n-1/2$		$\frac{3n+7}{9}-O(1/n)$	$n-1/2$	
Equitability	$\frac{(n+1)^2}{4n}$	n	$9/8$	∞	∞	2
	Divisible chores			Indivisible chores		
Proportionality	$\frac{(n+1)^2}{4n}$	n	$9/8$	n	n	2
Envy-freeness	$\frac{(n+1)^2}{4n}$	∞		∞	∞	
Equitability	n	n	2	∞	∞	∞

Table 1. Summary of our results (lower and upper bounds).

2 Fair division with goods

In this section, we focus on fair division and goods. We begin by presenting our results for the case of divisible goods.

Theorem 1. *For n players and divisible goods, the price of proportionality is $\Theta(\sqrt{n})$.*

Proof. Consider an instance with n players and let $\mathcal{O}$ denote the optimal allocation and OPT be the total utility of $\mathcal{O}$. We partition the set of players into two sets, namely L and S, so that if a player obtains utility at least $1/\sqrt{n}$ in $\mathcal{O}$, then she belongs to L, otherwise she belongs to S. Clearly, $OPT < |L| + |S|/\sqrt{n}$. We now describe how to obtain a proportional allocation $\mathcal{A}$; we distinguish between two cases depending on $|L|$.

We first consider the case $|L| \geq \sqrt{n}$; hence, $|S| \leq n - \sqrt{n}$. Then, for any negligibly small item that is allocated to a player $i \in L$ in $\mathcal{O}$, we allocate to i a fraction of $\sqrt{n}/n$ of the item, while we allocate to each player $i \in S$ a fraction of $\frac{n-\sqrt{n}}{n|S|} \geq 1/n$. Furthermore, for any negligibly small item that is allocated to a player $i \in S$ in $\mathcal{O}$, we allocate to each player $i \in S$ a fraction of $1/|S| > 1/n$. In this way, all players obtain a utility of at least $1/n$, while all items are fully allocated; hence, $\mathcal{A}$ is proportional. For every player $i \in L$, her utility in $\mathcal{A}$ is exactly $1/\sqrt{n}$ times her utility in $\mathcal{O}$, while every player $i \in S$ obtained a utility strictly less than $1/\sqrt{n}$ in $\mathcal{O}$ and obtains utility at least $1/n$ in $\mathcal{A}$. So, we conclude that the total utility in $\mathcal{A}$ is at least $1/\sqrt{n}$ times the optimal total utility.

Otherwise, let $|L| < \sqrt{n}$. Since $OPT < |L| + |S|/\sqrt{n}$, we obtain that $OPT < 2\sqrt{n} - 1$, while the total utility of any proportional allocation is at least 1. Hence, in both cases we obtain that the price of proportionality is $O(\sqrt{n})$. We continue by presenting a lower bound of $\Omega(\sqrt{n})$.

Consider the following instance with n players and $m < n$ items. Player i, for $i = 1, ..., m$, has utility 1 for item i and 0 for any other item, while player i, for $i = m+1, ..., n$, has utility $1/m$ for any item. In the optimal allocation, item i, for $i = 1, ..., m$, is allocated to player i, and the total utility is m. Consider any proportional allocation and let x be the sum of the fractions of the items that are allocated to the last $n - m$ players. The total utility of these players is x/m. Clearly, $x \geq m(n-m)/n$, otherwise some of them would obtain a utility less than $1/n$ and the allocation would not be proportional. The first m players are allocated the remaining fraction of $m - x$ of the items and their total utility is at most $m - x$. The total utility of all players is $m - x + x/m \leq \frac{m^2+n-m}{n}$. We conclude that the price of proportionality is at least $\frac{mn}{m^2+n-m}$ which becomes more than $\sqrt{n}/2$ by setting $n = m^2$. □

For the price of equitability, we can show that when the number of players is large, equitability may provably lead to less efficient allocations.

Theorem 2. *For n players and divisible goods, the price of equitability is at most n and at least $\frac{(n+1)^2}{4n}$.*

Since every envy-free allocation is also proportional, the lower bound on the price of proportionality also holds for envy-freeness. Interestingly, in the case of two players, there always exist almost optimal proportional or equitable allocations. Recall that in this case proportionality and envy-freeness are equivalent.

Theorem 3. *For two players and divisible goods, the price of proportionality (or envy-freeness) is $8 - 4\sqrt{3} \approx 1.072$, and the price of equitability is $9/8$.*

Proof. We only present the result for proportionality here. Consider an optimal allocation $\mathcal{O}$ and a proportional allocation $\mathcal{E}$ that maximizes the total utility of the players. We partition the cake into four parts A, B, C, and D: A is the part of the cake which is allocated to player 1 in both $\mathcal{O}$ and $\mathcal{E}$, B is the part which is allocated to player 2 in both $\mathcal{O}$ and $\mathcal{E}$, C is the part which is allocated to player 1 in $\mathcal{O}$ and to player 2 in $\mathcal{E}$, and D is the part of the cake which is allocated to player 1 in $\mathcal{E}$ and to player 2 in $\mathcal{O}$. In the following, we use the notation $u_i(X)$ to denote the utility of player i for part X of the cake.

Since $\mathcal{O}$ maximizes the total utility, we have $u_1(A) \geq u_2(A)$, $u_1(B) \leq u_2(B)$, $u_1(C) \geq u_2(C)$, and $u_1(D) \leq u_2(D)$. First observe that if $u_1(C) = u_2(C)$ and $u_1(D) = u_2(D)$, then $\mathcal{E}$ has the same total utility with $\mathcal{O}$. So, in the following we assume that this is not the case.

We consider the case $u_1(C) > u_2(C)$; the other case is symmetric. In this case, we also have that $u_1(D) = u_2(D) = 0$. Assume otherwise that $u_2(D) > 0$.

Then, there must be a subpart X of C for which player 1 has utility x and player 2 has utility at most $x \cdot u_2(C)/u_1(C)$ and a subpart Y of D for which player 2 has utility x and player 1 has utility at least $x \cdot u_1(D)/u_2(D)$. Then, the allocation in which player 1 gets parts A, X, and $D-Y$ and player 2 gets parts B, $C-X$, and Y is proportional and has larger utility than $\mathcal{E}$.

Now, we claim that $u_2(A) = 1/2$. Clearly, since $\mathcal{E}$ is proportional, the utility of player 2 in $\mathcal{E}$ is at least $1/2$, i.e., $u_2(B) + u_2(C) \geq 1/2$. Since the utilities of player 2 sum up to 1 over the whole cake, we also have that $u_2(A) \leq 1/2$. If it were $u_2(A) < 1/2$, then we would have $u_2(B) + u_2(C) > 1/2$. Then, there would exist a subpart X of C for which player 2 has utility x for some $x \leq 1/2 - u_2(A)$ and player 1 has utility larger than x. By allocating X to player 1 instead of player 2, we would obtain another proportional allocation with larger total utility.

Also, it holds that $u_2(A)/u_1(A) \leq u_2(C)/u_1(C)$. Otherwise, there would exist a subpart X of C for which player 1 has utility x and player 2 has utility $u_2(X)$ at most $x \cdot u_2(C)/u_1(C)$ and a subpart Y of A for which player 1 has utility x and player 2 has utility $u_2(Y)$ at least $x \cdot u_2(A)/u_1(A) > x \cdot u_2(C)/u_1(C) \geq u_2(X)$. By allocating the subpart X to player 1 and subpart Y to player 2, we would obtain another proportional allocation with larger total utility.

By the discussion above, we have $u_2(C) \geq \frac{u_1(C)}{2u_1(A)}$. We are now ready to bound the ratio of the total utility of $\mathcal{O}$ over the total utility of $\mathcal{E}$ which will give us the desired bound. We obtain that the price of proportionality is

$$\begin{aligned}\frac{u_1(A) + u_2(B) + u_1(C)}{u_1(A) + u_2(B) + u_2(C)} &= \frac{u_1(A) + 1/2 - u_2(C) + u_1(C)}{u_1(A) + 1/2} \\ &\leq \frac{u_1(A) + 1/2 + u_1(C)\left(1 - \frac{1}{2u_1(A)}\right)}{u_1(A) + 1/2} \\ &\leq \frac{u_1(A) + 1/2 + (1 - u_1(A))\left(1 - \frac{1}{2u_1(A)}\right)}{u_1(A) + 1/2}\end{aligned}$$

where the last inequality follows since $u_1(A) \geq u_2(A) = 1/2$ and $u_1(C) \leq 1 - u_1(A)$. The last expression is maximized to $8 - 4\sqrt{3}$ for $u_1(A) = \frac{1+\sqrt{3}}{4}$ and the upper bound follows.

In order to prove the lower bound, it suffices to consider a cake consisting of two parts A and B. Player 1 has utility $u_1(A) = 1$ and $u_1(B) = 0$ and player 2 has utility $u_2(A) = \sqrt{3} - 1$ and $u_2(B) = 2 - \sqrt{3}$. □

Moreover, it is easy to show an upper bound of $n - 1/2$ for the price of envy-freeness for both divisible and indivisible goods.

We next present our results that hold explicitly for indivisible goods; these results are either exact or tight within a constant factor.

Theorem 4. *For n players and indivisible goods, the price of proportionality is $n - 1 + 1/n$.*

Proof. We begin by proving the upper bound. Consider an instance and a corresponding optimal allocation. If this allocation is proportional, then the price of proportionality is 1; assume otherwise. In any proportional allocation, each player has utility at least $1/n$ on the pieces of the cake she receives and the total utility is at least 1. Since the optimal allocation is not proportional, some player has utility less than $1/n$ and the total utility in the optimal allocation is at most $n-1+1/n$.

We now present the lower bound. Consider the following instance with n players and $2n-1$ items. Let $0<\epsilon<1/n$. For $i=1,...,n-1$, player i has utility ϵ for item i, utility $1-1/n$ for item $i+1$, utility $1/n-\epsilon$ for item $n+i$ and utility 0 for all other items. The last player has utility $1/n-\epsilon$ for items $1,2,...,n-1$, utility $1/n+(n-1)\epsilon$ for item n, and utility 0 for all other items.

We argue that the only proportional allocation assigns items i and $n+i$ to player i for $i=1,...,n-1$, and item n to player n. To see that, notice that each player must be allocated at least one of the first n items, regardless of what other items she obtains, in order to be proportional. Since there are n players, each of them must be allocated exactly one of the first n items. Now, consider player n. It is obvious that she must be allocated item n, since she has utility strictly less than $1/n$ for any other item. The only available items (with positive utility) left for player $n-1$ are items $n-1$ and $2n-1$, and it is easy to see that both of them must be allocated to her. Using the same reasoning for players $n-2, n-3, ..., 1$, we conclude that the only proportional allocation is the aforementioned one, which has total utility $1+(n-1)\epsilon$.

Now, the total utility of the optimal allocation is lower-bounded by the total utility of the allocation where player i gets items $i+1$ and $n+i$, for $i=1,...,n-1$, and player n gets the first item. The total utility obtained by this allocation is $(1-1/n+1/n-\epsilon)(n-1)+\frac{1}{n}-\epsilon = n-1+1/n-n\epsilon$. By selecting ϵ to be arbitrarily small, the theorem follows. □

The above lower bound construction uses instances with no envy-free allocation and, hence, the lower bound on the price of proportionality does not extend to envy-freeness. We have a slightly weaker lower bound on the price of envy-freeness for indivisible goods which uses a more involved construction.

Theorem 5. *For n players and indivisible goods, the price of envy-freeness is at least $\frac{3n+7}{9}-O(1/n)$.*

Unfortunately, equitability may lead to arbitrarily inefficient allocations of indivisible goods when the number of players is at least 3.

Theorem 6. *For n players and indivisible goods, the price of equitability is 2 for $n=2$ and infinite for $n>2$.*

3 Fair division with chores

Our next theorem considers divisible chores.

Theorem 7. *For n players and divisible chores, the price of proportionality is at most n and at least $\frac{(n+1)^2}{4n}$, and the price of equitability is n.*

Since every envy-free allocation is also proportional, the lower bound on the price of proportionality also holds for envy-freeness. We also have a matching upper bound of $9/8$ for proportionality (or envy-freeness) in the case $n = 2$.

Finally, we consider the case of indivisible chores. Although the price of proportionality is bounded, the price of envy-freeness and equitability is infinite.

Theorem 8. *For n players and indivisible chores, the price of proportionality is n, whereas the price of envy-freeness (for $n \geq 3$) and equitability (for $n \geq 2$) is infinite.*

Proof. Due to lack of space, we only present the case of envy-freeness. Consider the following instance with n players and $2n$ items. Let $\epsilon < 1/(2n)$. For $i = 1, \ldots, n-2$, player i has disutility $1/n$ for the first n items and disutility 0 for every other item. Player $n-1$ has disutility 0 for the first $n-1$ items, disutility ϵ for item n, disutility $1/n$ for items $n+1, \ldots, 2n-1$ and disutility $1/n - \epsilon$ for item $2n$. Finally, player n has disutility 0 for the first $n-1$ items, disutility $1/(2n)$ for items n and $2n$, and disutility $1/n$ for items $n+1, \ldots, 2n-1$.

Clearly, the optimal allocation has total disutility ϵ and is obtained by allocating items $n+1, \ldots, 2n$ to players $1, \ldots, n-2$, item n to player $n-1$, and items $1, \ldots, n-1$ either to player $n-1$, or to player n. In each case, player $n-1$ envies player n. Furthermore, the allocation in which player i, for $i = 1, \ldots, n$ is allocated items i and $i+n$ is envy-free. The remark that concludes this proof is that there cannot exist an envy-free allocation having negligible disutility (i.e., less than $1/(2n)$). □

References

1. E. Anshelevich, A. Dasgupta, J. M. Kleinberg, E. Tardos, T. Wexler, and T. Roughgarden. The price of stability for network design with fair cost allocation. *SIAM Journal of Computing*, Vol. 38(4), pp. 1602–1623, 2008.
2. N. Bansal and M. Sviridenko. The Santa Claus problem. In *Proceedings of the 38th Annual ACM Symposium on Theory of Computing (STOC '06)*, pp. 31–40, 2006.
3. S. J. Brams and A. D. Taylor. Fair division: From cake-cutting to dispute resolution. *Cambridge University Press*, 1996.
4. J. Edmonds and K. Pruhs. Cake-cutting really is not a piece of cake. In *Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '06)*, pp. 271–278, 2006.
5. A. Kumar and J. Kleinberg. Fairness measures for resource allocation. In *Proceedings of the 41st Annual Symposium on Foundations of Computer Science (FOCS '00)*, pp. 75–85, 2000.
6. R. Lipton, E. Markakis, E. Mossel and A. Saberi. On approximately fair allocations of indivisible goods. In *Proceedings of the 5th ACM Conference on Electronic Commerce (EC '04)*, pp. 125–131, 2004.
7. A. Procaccia. Thou shalt covet thy neighbor's cake. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence (IJCAI '09)*, 2009.
8. G. J. Woeginger and J. Sgall. On the complexity of cake-cutting. *Discrete Optimization*, Vol. 4, pp. 213–220, 2007.