On low-envy truthful allocations*

Ioannis Caragiannis, Christos Kaklamanis, Panagiotis Kanellopoulos, and Maria Kyropoulou

Research Academic Computer Technology Institute and
Department of Computer Engineering and Informatics
University of Patras, 26500 Rio, Greece

Abstract. We study the problem of allocating a set of indivisible items to players having additive utility functions over the items. We consider allocations in which no player envies the bundle of items allocated to the other players too much. We present a simple proof that deterministic truthful allocations do not minimize envy by characterizing the truthful mechanisms for two players and two items. Also, we present an analysis for uniformly random allocations which are naturally truthful in expectation. These results simplify or improve previous results of Lipton et al.

1 Introduction

Resource allocation [9] has been an important problem in several areas such as Computer Science, Artificial Intelligence, and Economics since their early days. In the era of the Internet with a vast amount of computational, communication, and storage resources available worldwide, the problem is still of paramount importance. Besides efficiency, fairness is another important aspect that resource allocation must satisfy. Additional constraints such as the selfish behavior of resource owners and users make the variations of the problem very challenging.

A simple but foundational resource allocation problem is the well-known *cake-cutting* problem [6, 20]. In cake-cutting, we are given n players with different utilities for different parts of a cake. The objective is to allocate pieces of the cake to the players in such a way that they are satisfied. Traditionally, satisfaction of players has been measured by two different notions: *envy-freeness* and *proportionality*. Envy-freeness means that each player prefers her allocated pieces to the pieces allocated to any other player. Proportionality means that the utility of each player for the pieces allocated to her is at least $1/n$ times her utility for the whole cake. Due to the continuity of the cake and the utilities of the players, both objectives are always feasible.

A similar problem concerns the fair allocation of *indivisible items*; indivisibility implies that an item cannot be broken into parts and must be allocated to

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a single player. Here, we again have a set $\mathcal{N}$ of n players and a set $\mathcal{M}$ of m indivisible items. Each player p has a non-negative utility function $u_p : 2^{\mathcal{M}} \rightarrow \mathbb{R}_0^+$. The objective is to assign to each player p a bundle of items $\mathcal{M}_p \subseteq \mathcal{M}$, so that $\bigcup_p \mathcal{M}_p = \mathcal{M}$ and some criterion concerning fairness is maintained. An important special case is that of *additive utilities*. In this case, each player p has a utility $u_{p,i}$ for each item $i \in \mathcal{M}$ and her utility for a bundle of items is simply the sum of her utilities on these items. In contrast to the cake-cutting problem, envy-freeness and proportionality are not always feasible goals in this setting even in the case of additive utilities. Here, *envy minimization* is among the most prominent measures of fairness. Given an allocation A in which players p and q are assigned bundles $\mathcal{M}_p$ and $\mathcal{M}_q$, the envy $e_{pq}(A)$ of player p for player q is $e_{pq}(A) = u_p(\mathcal{M}_q) - u_p(\mathcal{M}_p)$. Then, envy of A is defined as $e(A) = \max_{p,q \in \mathcal{N}} e_{pq}$. Clearly, A is envy-free if $e(A) = 0$.

An implicit assumption in the above definitions is that the players express their true utilities which are used by the algorithm (i.e., the allocation function) in order to compute an allocation. In practice, players are usually *selfish* in the sense that they aim to increase their benefit, i.e., their total utility on the bundle of items the algorithm allocates to them. In order to do so, they may report false *valuations* of items to the algorithm (i.e., different than their true utilities). *Truthful allocation functions* guarantee that the allocation is based on the true utilities of the players. A deterministic allocation function is *truthful* if the benefit obtained by a player when reporting false valuations on the items is not greater than the benefit she would have obtained by telling the truth. Similarly, a randomized allocation function is *truthful in expectation* if the expected benefit of a player is maximized when revealing her true utilities.

Related work. Research concerning fair allocations originated in the 1940s with a focus on cake-cutting [21]. Since then, the problem of achieving a proportional allocation with the minimum number of operations has received much attention and is now well-understood [12, 13, 6, 20, 24]. The problem of achieving envy-freeness has been proven to be much more challenging [8, 5, 22]; in fact, under the most common computational model of cut and evaluation queries [20], no algorithm with bounded running time is known for more than 3 players. Very recently, envy-freeness was proved to be a harder property to achieve than proportionality [19, 23]. Better solutions exist for different computational models (e.g., moving knife algorithms [7]).

Lipton et al. [16] studied envy minimization with indivisible items. Among other results, they proved that allocations with envy bounded by the *marginal utility* always exist and can be computed in polynomial time. In the case of additive utilities, marginal utility translates to the maximum per item utility over all players. They also present algorithms that compute allocations that approximate the minimum *envy-ratio*; the envy ratio of a player p for a player q is the utility of player p for the items allocated to player q over p's utility for the items allocated to her. Complexity considerations about envy-freeness for indivisible items and non-additive utilities are presented in [4]. The papers [10, 11] study the problem of achieving envy-free and efficient allocations in distributed settings and when the allocation of items is accompanied by monetary side payments (in this case, envy-freeness is always a feasible goal). Lipton et al. [16] also consider truthful allocations; they show that any deterministic allocation function that returns an allocation with minimum possible envy cannot be truthful; their proof uses an instance with two players and many items. Finally, they present an analysis of the randomized allocation function that assigns each item to one of the players uniformly at random and independently of the allocations of the other items. This allocation function is truthful in expectation. For the case where the sum of utilities of each player over the items is 1, they prove that, with high probability, the envy of the resulting allocation is $O(\sqrt{\alpha}n^{1/2+\epsilon})$, where α is the maximum utility per item over all players and ϵ is an arbitrarily small positive number. We remark that the study of truthful allocation functions belongs to the recent line of research on *algorithmic mechanism design* [18]. In particular, Mu'alem and Schapira [17] prove lower bounds on the envy of truthful allocation functions. However, unlike the model of [16] which we also follow in the current paper, [17] and most of the studies in algorithmic mechanism design allow monetary transfers between the players.

For indivisible items, a fairness objective that has been extensively considered recently is *max-min fairness*. Here, the objective is to compute an allocation in which the benefit of the least happy player is maximized. The problem was studied by Bezáková and Dani [3] and Golovin [14] who obtained approximation algorithms that provably return a solution that is always a factor of $O(n)$ within the optimal value. The problem was popularized by Bansal and Sviridenko [2] as the *Santa Claus problem*, where Santa Claus aims to distribute presents to the kids so as to maximize the happiness of the least happy kid. Subsequently, Asadpour and Saberi [1] presented an $O(\sqrt{n}\log^3 n)$-approximation algorithm for this problem.

Our results. In this paper, we consider allocation of indivisible items to players having additive utility functions over the items. We present an alternative proof that no deterministic truthful allocation function minimizes envy by characterizing the deterministic truthful allocation functions for two players and two items. Our proof actually shows that for any truthful allocation function, there are instances in which the envy is almost maximized. Our proof simplifies the proof of Lipton et al. [16] that uses a large number of items. Our impossibility result trivially extends to the case of many players and many items and also to the more general case of non-additive utility functions. We also present an improved analysis of uniformly random allocations of m items over n players. We show that the envy is at most $O(\alpha\sqrt{m \ln n})$ with high probability, where α is the maximum utility per item over all players and items. For the case where the sum of utilities of each player is 1, we prove a bound of $O(\sqrt{\alpha \ln n})$. This improves the previous bound of $O(\sqrt{\alpha}n^{1/2+\epsilon})$ for any $\epsilon > 0$ [16]. Our proof follows similar lines to the proof of [16] but we exploit the fact that the allocation of each item is independent and use the Hoeffding bound instead of the Chebychev inequality in order to bound the envy.

Roadmap. Our characterization of the deterministic truthful allocations for two players and two items is presented in Section 2. The analysis of random allocations is presented in Section 3.

2 Truthful allocations for two players and two items

In this section we present a characterization of deterministic truthful allocations with two players and two items.

In general, the first player will have utilities $u_1 x$ for the first item and $u_1(1-x)$ for the second one while the utilities of the second player are $u_2 y$ and $u_2(1-y)$, respectively. Here, $x, y \in [0, 1]$ and u_1, u_2 are the sums of utilities of the two players for both items. So, an allocation function gets as input u_1, u_2, x, and y and computes an allocation of the items to the players. We denote each of the four possible allocations as a 2×2 matrix with entries 1 and 0. The columns correspond to the players and the rows to the items. An 1 in an entry of such a matrix indicates that the item corresponding to the row is allocated to the player corresponding to the column.

We use the term non-boundary values to denote real numbers in $[0, 1]$ different than 0, $1/2$, and 1. We consider only non-boundary values for x and y since they suffice for proving our main result on the envy. Our characterization can be easily extended to boundary values of x and y as well.

We begin with an observation that simplifies the allocation functions that have to be considered.

Lemma 1. *For non-boundary values of x and y, no truthful allocation function f depends on u_1 and u_2.*

Proof. Assume that this is not the case and that f computes different allocations on inputs (u_1, u_2, x, y) and (u_1', u_2, x, y) where x, y have non-boundary values and $u_1 \neq u_1'$.

When x has a non-boundary value, the four different possible allocations $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ yield different benefit to player 1 when her utilities on the items are $u_1 x$ and $u_1(1-x)$, namely u_1, $u_1 x$, $u_1(1-x)$, and 0.

Now assume that the function f returns an allocation of higher benefit to player 1 when she reports u_1' instead of u_1. Then, player 1 has an incentive to lie. If this is not the case and f returns an allocation of lower benefit when player 1 reports u_1', then consider the case when player 1 has true utilities $u_1' x$ and $u_1'(1-x)$ on the two items. In this case, player 1 would have an incentive to lie and report $u_1 x$ and $u_1(1-x)$ as her valuation. □

By Lemma 1, we may assume that f depends only on x and y when they have non-boundary values. Without loss of generality, we also assume that $u_1 = u_2 = 1$ in the following.

Lemma 2. *A truthful allocation function f has the following properties:*

(a) If $f(x^*, y^*)$ assigns both items to the same player for some non-boundary values x^*, y^*, then $f(x, y)$ assigns both items to that player for any non-boundary values x, y.
(b) If $f(x^*, y^*)$ assigns to player 1 the item which she prefers the least for some non-boundary values x^*, y^*, then $f(x, y^*)$ assigns that item to player 1 for any non-boundary value x.
(c) If $f(x^*, y^*)$ assigns to player 2 the item which she prefers the least for some non-boundary values x^*, y^*, then $f(x^*, y)$ assigns that item to player 2 for any non-boundary value y.

Proof. (a) Assume that the allocation function assigns both items to player 1 for some non-boundary values and at most one of the items for some other non-boundary values. Then, one of the following must hold:

– There exist non-boundary values x^*, y^*, x' such that $f(x^*, y^*)$ assigns both items to player 1 and $f(x', y^*)$ assigns at most one of the items to player 1. In this case, if the true utility of player 1 for item 1 is x', she has an incentive to lie and report x^* in order to get both items.
– There exist non-boundary values x^*, y^*, y' such that $f(x^*, y^*)$ assigns both items to player 1 and $f(x^*, y')$ assigns at most one of the items to player 1. In this case, if the true utility of player 2 for item 1 is y^*, she has an incentive to lie and report y' in order to get at least one item.

The case in which the allocation function assigns both items to player 2 is symmetric.

(b) Consider the case with $x^* < 1/2$ (the case $x^* > 1/2$ is symmetric) so that $f(x^*, y^*)$ assigns item 1 to player 1. Assume otherwise that there exists a non-boundary value x' such that $f(x', y^*)$ assigns item 2 to player 1. Then, if the true utility of player 1 for item 1 is x^*, player 1 has an incentive to lie and report x' in order to get item 2 which she prefers the most.

(c) The proof of this case is very similar to (b). □

The properties of Lemma 2 yield the following.

Lemma 3. *The only truthful allocations with respect to non-boundary item valuations are those depicted in Figure 1.*

Proof. Figure 1 contains the eight possible allocation functions that satisfy the properties of Lemma 2. Truthfulness follows since for each player, given the valuation of the other player, these allocation functions either assign her the most preferred item or the allocation does not depend on her valuations. □

We are now ready to prove the main statement of this section.

Theorem 1. *No truthful allocation minimizes envy.*

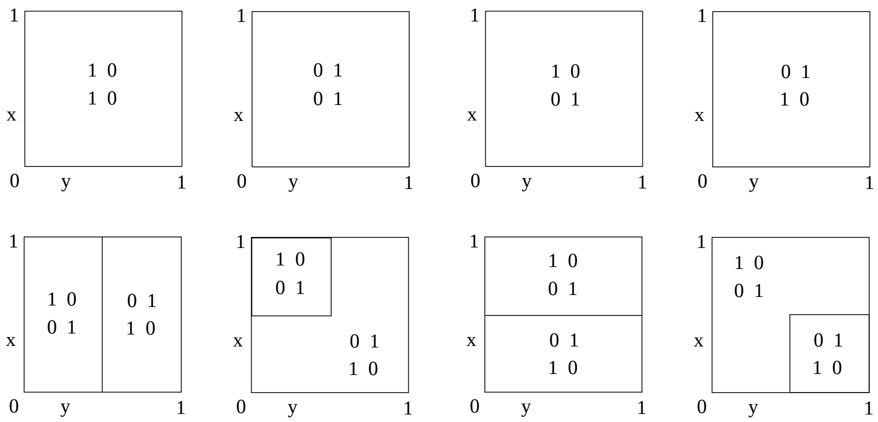

Fig. 1. The eight truthful allocation functions for two players.

Proof. Clearly, for the two first fixed truthful allocation functions of Figure 1, both items are assigned to one player and hence the other player always has envy 1. Let $\epsilon \in (0, 1/4)$. Consider the two valuation pairs $(1-\epsilon, 1/2+\epsilon)$ and $(1/2+\epsilon, 1-\epsilon)$. The allocations $\begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix}$ and $\begin{pmatrix} 0\,1 \\ 1\,0 \end{pmatrix}$ yield envy 2ϵ, respectively. For any of the six last truthful allocation functions of Figure 1, in one of these valuations pairs, the allocation yields benefit ϵ for one player and $1/2+\epsilon$ to the other. Hence, one player has envy $1-2\epsilon$. By setting ϵ very close to 0, we have that the envy is actually maximized. □

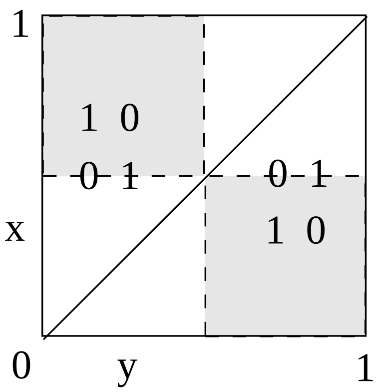

Fig. 2. The minimum envy allocation function when the sum of utilities of each player on the two items is 1.

We remark that the four non-fixed allocation functions in Figure 1 produce an envy-free allocation if one exists. Figure 2 presents the allocations that minimize envy when the sum of the utilities of each player on the items is 1. The grey areas indicate the cases of envy-free allocations. The two pairs of valuations considered in the proof of Theorem 1 have been selected to be outside but very close to these areas.

3 Improved analysis of random allocations

In this section we consider the randomized allocation function that allocates each item to a player selected uniformly at random among the n players and in such a way that the allocation of an item is independent of the other allocations. Note that the allocation function does not depend on the valuations of the players. Hence, it is truthful in expectation since no player has an incentive to report a false valuation in order to increase her expected benefit. We present an upper bound on the envy of the resulting allocations using Hoeffding inequality [15].

Theorem 2 (Hoeffding [15]). *Let $X_1, \dots, X_k$ be independent random variables with $\Pr(X_i \in [a_i, b_i]) = 1$ for $1 \le i \le k$. Then, for the sum of these variables $S = \sum_{i=1}^{k} X_i$, we have*

$$\Pr(S - \mathbb{E}[S] \ge t) \le \exp\left(-\frac{2\,t^2}{\sum_{i=1}^{k}(b_i - a_i)^2}\right).$$

So, the particular version of Hoeffding inequality upperbounds the probability that a random variable which can be expressed as the sum of independent random variables exceeds its expectation by a certain amount. Our statement is the following; besides the number of players and items, it is also expressed in terms of the maximum utility per item over all players. We assume that the number n of players is large and the term high probability denotes a probability of $1 - 1/n$. We also denote by $v_{p,i}$ the utility of player p for item i.

Theorem 3. *Consider an instance with n players and m items and let $\alpha = \max_{p,i} v_{p,i}$.*

(a) With high probability, the random allocation yields an envy of at most $O(\alpha\sqrt{m \ln n})$.
(b) If the sum of utilities of each player is 1, then with high probability, the random allocation yields an envy of at most $O(\sqrt{\alpha \ln n})$.

Proof. Consider two players p and q. We define the random variable Y_i^{pq} indicating the contribution of item i to the envy of player p for player q. Then, the envy S^{pq} of player p for player q is $S^{pq} = \sum_{i=1}^{m} Y_i^{pq}$. Observe that

- $Y_i^{pq} = v_{p,i}$ if item i is allocated to q (and this happens with probability $1/n$),
- $Y_i^{pq} = -v_{p,i}$ if item i is allocated to p (and this happens with probability $1/n$), and

– $Y_i^{pq} = 0$ if item i is not allocated to p or q (this happens with probability $1 - 2/n$).

Clearly, the random variables Y_i^{pq} are independent, $\Pr(Y_i^{pq} \in [-v_{p,i}, v_{p,i}]) = 1$ and $\mathbb{E}[S^{pq}] = 0$. By applying the Hoeffding bound for any $t \geq 0$, we have

$$\Pr(S^{pq} \geq t) \leq \exp\left(-\frac{t^2}{2\sum_{i=1}^{m} v_{p,i}^2}\right). \quad (1)$$

In order to prove (a), we use inequality (1) by setting $t = \alpha\sqrt{6m \ln n}$ and the fact that $\sum_{i=1}^{m} v_{p,i}^2 \leq m\alpha^2$ to obtain

$$\Pr(S^{pq} \geq \alpha\sqrt{6m \ln n}) \leq 1/n^3.$$

Since there are at most n^2 pairs of players p, q, by applying the union bound we have that the probability that the maximum envy between any two players exceeds $\alpha\sqrt{6m \ln n}$ is at most $1/n$.

In order to prove (b), we set $t = \sqrt{6\alpha \ln n}$ and use the fact that $\sum_{i=1}^{m} v_{p,i}^2 \leq \alpha$ when $\sum_{i=1}^{m} v_{p,i} = 1$ and $v_{p,i} \geq 0$. By (1), we obtain that

$$\Pr(S^{pq} \geq \sqrt{6\alpha \ln n}) \leq 1/n^3.$$

Again, by applying the union bound we have that the probability that the maximum envy between any two players exceeds $\sqrt{6\alpha \ln n}$ is at most $1/n$. □

The first upper bound should be compared to the lower bound of α [16] on the envy of allocations in which the maximum utility per item among all players is α. This bound is shown to be tight in [16] but the upper bound is not obtained by a truthful allocation function. Our second upper bound significantly improves the upper bound of $O(\sqrt{\alpha} n^{1/2+\epsilon})$ for the case where the sum of utilities of each player is 1. Whether there exist better allocation functions (i.e., that yield allocations with smaller envy) that are truthful in expectation is an interesting open problem.

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