New Bounds on the Size of the Minimum Feedback Vertex Set in Meshes and Butterflies

Ioannis Caragiannis [a] Christos Kaklamanis [a]
Panagiotis Kanellopoulos [a]

[a] *Computer Technology Institute and*
Department of Computer Engineering and Informatics
University of Patras, 26500 Rio, Greece
E-mail:{caragian, kakl, kanellop}@ceid.upatras.gr

Key words: feedback vertex set, interconnection networks, algorithms

---

1 Introduction

A *feedback vertex set* of a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ whose removal from G induces an acyclic graph $G' = (V', E')$ with $V' = V \backslash S$ and $E' = \{(u, v) \in E : u, v \in V'\}$. If the cardinality of S is the minimum possible, we call it a *minimum feedback vertex set* of G.

The problem of finding a minimum feedback vertex set of a graph has been proved to be NP–hard [4], although polynomial time solutions have been found for particular classes of graphs, e.g., reducible graphs [11], cocomparability and convex bipartite graphs [7], cyclically reducible graphs [13], and interval graphs [8]. For general graphs, the best known approximation algorithm has approximation ratio 2 (Bafna *et al.* [1]).

The problem has important applications to several different contexts. The first application was in combinatorial circuit design (see the discussion in [3]), while it also applies to operating systems where resource allocation mechanisms which can prevent deadlocks are desirable. This can be carried out by finding a feedback vertex set in the dependence graph of the tasks that want to use a particular resource and moving the corresponding tasks into a waiting queue so that the new dependence graph becomes acyclic (and deadlock–free) [13]. Clearly, in this situation, it is also important to move into the waiting queue the minimum possible number of tasks.

Other applications of the problem are in artificial intelligence (in the constraint satisfaction problem and Bayesian inference [2]), in the study of monopolies in synchronous distributed systems [10], and, recently, in the problem of converters placement in optical networks [5,12].

In this letter we give new bounds on the size of the minimum feedback vertex set in meshes and butterflies. A formal description of these graphs can be found in [6]. We present alternative ways for computing small feedback vertex sets in 2–dimensional meshes. We also present a method which extends to meshes of higher dimensions. We prove that the bounds we obtain for meshes are asymptotically optimal (provided that the size of the mesh is large) by giving almost matching lower bounds.

For butterflies, we obtain our upper bound by improving the analysis of an algorithm presented in [9]. Using a simple argument, we obtain an almost matching lower bound, thus proving that our upper bound is asymptotically tight (provided that the dimension of the butterfly is large).

The best previously known bounds for the problem in these particular classes of graphs were due to Luccio [9]. In the following table, we summarize our main results and relate them to previously known ones.

	Previous results (Luccio [9])		Our results	
	Upper bound	Lower bound	Upper bound	Lower bound
2-D mesh	$\frac{mn}{3}+\frac{m+n}{6}+o(m,n)$	$\frac{mn}{3}-\frac{n+m-2}{3}$	$\frac{mn}{3}-\frac{m+n-5}{6}$	
d-D mesh			$\frac{(d-1)n^d}{2d-1}$	$\frac{(d-1)n^d-dn^{d-1}+1}{2d-1}$
Butterfly	$\frac{11d\,2^d}{32}+d$	$2^{d-1}\lfloor\frac{d+1}{2}\rfloor$	$\frac{(d+1/2)2^d}{3}$	$\frac{(d-1)2^d+1}{3}$

The rest of the letter is structured as follows. In Section 2 we give a simple and general lower bound argument which yields lower bounds in any graph. Two alternative upper bounds for the 2–dimensional mesh are presented in Section 3. The upper bound for meshes of higher dimensions is presented in Section 4 while the upper bound for the butterfly is given in Section 5.

2 Lower bounds

Luccio in [9] proved lower bounds for 2-dimensional meshes, tori and butterflies. In this section we present a simple argument useful to prove lower bounds in any graph. Using this argument, we extend the known lower bounds for 2–dimensional meshes to meshes of higher dimensions and improve the known lower bound for the size of the minimum feedback vertex set in the butterfly.

Note that this argument is similar to the one recently used in [3] to prove a lower bound for the hypercube.

Our proofs are based on the following observation.

Fact 1 *A set $S \subseteq V$ is a feedback vertex set of G if and only if the subgraph of G induced by the vertices of $V \backslash S$ is a forest.*

Lemma 2 *Any feedback vertex set in a graph $G(V, E)$ with maximum degree r has size at least $\frac{|E|-|V|+1}{r-1}$.*

PROOF. Let $S \subseteq V$ be a feedback vertex set of G. The subgraph $H(V', E')$ of G induced by the vertices in $V' = V \backslash S$ has $|E'| \geq |E| - r|S|$ edges. By Fact 1, H is a forest; thus, we obtain that

$$\begin{aligned} |V'| &\geq |E'| + 1 \Rightarrow \\ |V| - |S| &\geq |E| - r|S| + 1 \Rightarrow \\ |S| &\geq \frac{|E| - |V| + 1}{r - 1} \qquad \square \end{aligned}$$

We can immediately apply Lemma 2 to meshes, tori, butterflies, and other graphs. We only state the lower bounds for the d–dimensional mesh and the d–dimensional butterfly here.

Corollary 3 *The minimum feedback vertex set of the d–dimensional mesh with $n \times n \times ... \times n = n^d$ nodes has size at least $\frac{(d-1)n^d - dn^{d-1}+1}{2d-1}$.*

Note that the d–dimensional mesh with $n \times n \times ... \times n = n^d$ nodes has maximum degree $2d$ and $dn^d - dn^{d-1}$ edges.

Corollary 4 *The minimum feedback vertex set of the d–dimensional butterfly has size at least $\frac{(d-1)2^d+1}{3}$.*

Note that the d–dimensional butterfly has $(d+1)2^d$ nodes, $d2^{d+1}$ edges and maximum degree 4. Corollary 4 improves the $2^{d-1}\lfloor\frac{d+1}{2}\rfloor$ lower bound presented in [9].

3 Upper bounds on the 2–dimensional mesh

For 2–dimensional meshes, Luccio in [9] shows a construction of a feedback vertex set on the $n \times m$ mesh which has size at most $\frac{mn}{3} + \frac{m+n}{6} + o(m, n)$. The construction in [9] computes an optimal feedback vertex set in the case where $n = m = 2^r + 1$ for some integer $r > 0$. Below, we show a way to improve the upper bound in [9] for the case of $n \times m$ meshes. We begin with a very simple construction.

Consider the partition of the nodes of a two dimensional mesh M in the three sets V_0, V_1, V_2 defined as $V_k = \{v_{ij} | i + j = k \bmod 3\}$ for $k = 0, 1, 2$. We observe that any of the sets V_0, V_1, and V_2 is a feedback vertex set of $M = (V(M), E(M))$, since, for any $k = 0, 1, 2$, the subgraph of M induced by the vertices in $V(M) \backslash V_k$ consists of disjoint paths. Since the sets V_0, V_1, V_2 are disjoint, at least one of them has size at most $\lfloor |V(M)|/3 \rfloor$. We obtain the following result.

Theorem 5 *For any integers $n, m > 0$, the size of the minimum feedback vertex set of a $n \times m$ mesh is at most $\left\lfloor \frac{mn}{3} \right\rfloor$.*

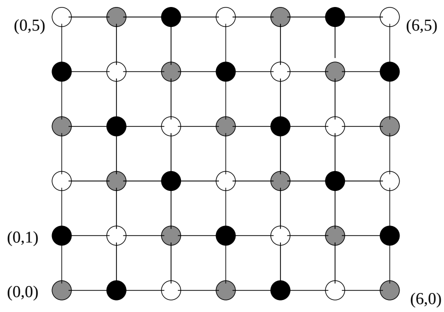

Fig. 1. Partition of the nodes of the 6×7 mesh into three disjoint feedback vertex sets.

An example of this construction is depicted in Fig. 1. Now, we show a way to improve Theorem 5.

Given a $n \times m$ mesh M, we apply the above idea to the lower right $(n-2) \times (m-2)$ submesh M'. Let $S(M')$ be the feedback vertex set of M', that consists of vertex-disjoint paths. By Theorem 5, we obtain that

$$|S(M')| \leq \frac{(n-2)(m-2)}{3} \tag{1}$$

Then, we consider the vertices in the boundary of M', i.e., the $n + m - 3$ vertices with coordinates

$$(1,0), (1,1), ..., (1, n-2), (2, n-2), ..., (m-2, n-2), (m-1, n-2)$$

Now we distinguish between three cases:

- Case 1: $(2, n-3) \in S(M')$. In this case, we define the set of vertices S' to contain the following vertices in the boundary:

$$(1, n-2), (3, n-2), ..., (m-1-(m \bmod 2), n-2) \qquad \text{and}$$

$$(1, n-4), (1, n-6), ..., (1, n \bmod 2)$$

i.e., vertex $(1, n-2)$ and every second vertex on the same row and column.

- Case 2: $(2, n-4) \in S(M')$. In this case, we define the set of vertices S' to contain the following vertices in the boundary:

$$(1, n-2), (3, n-2), ..., (m-1-(m \bmod 2), n-2) \qquad \text{and}$$

$$(1, n-3), (1, n-5), ..., (1, (n+1) \bmod 2)$$

i.e., vertex $(1, n-2)$ and every second vertex on the same row, and vertex $(1, n-3)$ and every second vertex downwards in the same column.

- Case 3: $(3, n-3) \in S(M')$. This case is similar to Case 2 and has a symmetric construction.

An example of this construction (for Case 2) is depicted in Fig. 2.

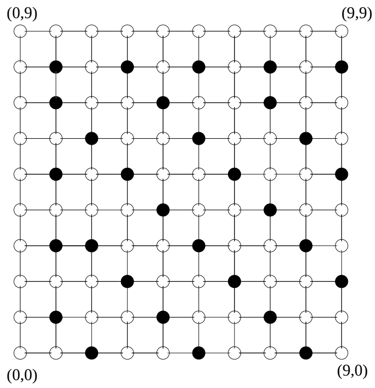

Fig. 2. The feedback vertex set for the 10×10 mesh.

It can be easily verified that

$$|S'| \leq \frac{n+m-1}{2} \tag{2}$$

The set $S(M') \cup S'$ is a feedback vertex set of M since its removal induces a subgraph of M which has a path consisting of the vertices with coordinates

$$(0,0), (0,1), ..., (0, n-1), (1, n-1), (2, n-1), ..., (m-1, n-1)$$

connected to disjoint paths. Clearly, the induced subgraph is a tree. Furthermore, using (1), (2) we have that

$$|S(M') \cup S'| \leq \frac{(n-2)(m-2)}{3} + \frac{n+m-1}{2} = \frac{nm}{3} - \frac{n+m-5}{6}$$

Theorem 6 *For any integer $n, m > 0$, the size of the minimum feedback vertex set of a $m \times n$ mesh is at most $\frac{nm}{3} - \frac{n+m-5}{6}$.*

4 Upper bound on meshes of higher dimension

Our construction of feedback vertex sets of small size in d–dimensional meshes ($d \geq 2$) is based on the existence of distance–3 independent sets of large size (a similar idea was used in [3] for constructing small feedback vertex sets in hypercubes).

Given a graph $G = (V, E)$, we call a set of nodes $T \subseteq V$ a *distance*–3 *independent set* if the distance between any two nodes in T is at least 3. An example on the 2–dimensional mesh is depicted in Fig. 3.

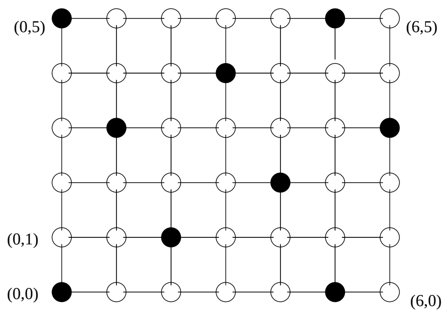

Fig. 3. A distance–3 independent set on the 6×7 mesh.

Lemma 7 *In any d-dimensional mesh $n \times n \times \cdots \times n$, there exists a distance–3 independent set of size at least $\frac{n^d}{2d+1}$.*

PROOF. Consider the partition of the n^d nodes of the mesh into $2d + 1$ sets of nodes T_0, T_1, ..., T_{2d} such that set T_k contains the nodes with coordinates $(x_1, x_2, ..., x_d)$ such that $\sum_{i=1}^{d} i \cdot x_i = k \pmod{2d+1}$. Clearly, the sets $T_0, T_1, ..., T_{2d}$ are mutually disjoint. We will show that each one of these sets is a distance–3 independent set of the mesh. In this way, we will obtain the lemma.

Consider two nodes u and v in T_k (for some k such that $0 \leq k \leq 2d$) with coordinates $u = (x_1, x_2, ..., x_d)$ and $v = (y_1, y_2, ..., y_d)$, respectively. If u and v differ in at least three coordinates, the distance between u and v is at least 3.

Assume that u and v differ in two coordinates, i and j, i.e, $x_i \neq y_i$, $x_j \neq y_j$, and $x_m = y_m$ for $m \neq i, j$. Since u and v belong to T_k, it holds that

$$\sum_{i=1}^{d} i \cdot x_i = k \pmod{2d+1} \qquad \text{and}$$

$$\sum_{i=1}^{d} i \cdot y_i = k \pmod{2d+1}.$$

We obtain that $i \cdot (x_i - y_i) + j \cdot (x_j - y_j) = 0 \pmod{2d+1}$. Now, if $|x_i - y_i| \geq 2$ or $|x_j - y_j| \geq 2$, the distance between u and v is at least 3. Assume that $|x_i - y_i| = 1$ and $|x_j - y_j| = 1$ which yields $i \pm j = 0 \pmod{2d+1}$, a contradiction since $i \neq j$ and $1 \leq i, j \leq d$.

Finally assume that u and v differ in the i-th coordinate, i.e., $x_i \neq y_i$, and $x_m = y_m$ for $m \neq i$. Then, $i \cdot (x_i - y_i) = 0 \pmod{2d+1}$. Now, if $|x_i - y_i| = 1$ or $|x_i - y_i| = 2$, we obtain that $i > d$, a contradiction. Thus, the distance between u and v is at least 3. $\square$

Now, we will show how to construct a feedback vertex set in a d–dimensional mesh using the construction of a distance–3 independent set presented in Lemma 7.

Theorem 8 *For any integers $n > 0$ and $d \geq 2$, the minimum feedback vertex set of a d–dimensional mesh with $n \times n \times ... \times n = n^d$ nodes has size at most $\frac{d-1}{2d-1}\ n^d$.*

PROOF. Given a d–dimensional mesh G_d with $n \times n \times ... \times n = n^d$ nodes, we construct a feedback vertex set as follows. We first apply Lemma 7 to obtain a distance–3 independent set T in the $(d-1)$–dimensional mesh G_{d-1} of size

$$|T| \geq \frac{n^{d-1}}{2d-1} \tag{3}$$

Now, consider the set $V' \subseteq V$ of vertices which have the first $d-1$ coordinates identical to the $d-1$ coordinates of some vertex in T. Clearly,

$$|V'| = n|T| \tag{4}$$

Consider the two (disjoint) subsets of $V \backslash V'$:

$$S_1 = \left\{ v = (x_1, x_2, ..., x_d) \in V \backslash V' : \sum_{i=1}^{d} x_i = 0 \pmod 2 \right\}$$

$$S_2 = \left\{ v = (x_1, x_2, ..., x_d) \in V \backslash V' : \sum_{i=1}^{d} x_i = 1 \pmod 2 \right\}.$$

Since S_1 and S_2 are disjoint, it holds that

$$\min\{|S_1|, |S_2|\} \leq \frac{|V \backslash V'|}{2} \tag{5}$$

Using (3), (4), and (5), we obtain that

$$\min\{|S_1|, |S_2|\} \leq \frac{d-1}{2d-1} n^d.$$

It remains to prove that both S_1 and S_2 are feedback vertex sets in G_d. Consider the set S_1 (the proof for S_2 is similar). Removing the vertices in S_1 from G_d induces a graph H with a set of vertices $S_2 \cup V'$. Clearly, S_2 is an independent set of H, i.e., no two vertices of S_2 are adjacent. Furthermore, by construction, the vertices in V' consitute lines; a line for each vertex of G_{d-1} which belongs to T. Thus, if there exists a cycle in H, then there must be at least a path in H between vertices belonging to different lines. Note that, by our construction, the distance between two vertices v_1, v_2 belonging to different lines, is at least 3. Therefore, any path between v_1 and v_2 must contain vertices of both S_1 and S_2. This contradicts the assumption that H is the subgraph induced from G_d by removing the vertices in S_1. This completes the proof of the theorem. $\square$

5 Upper bound on the butterfly

The upper bound of [9] on the size of the minimum feedback vertex set of the d-dimensional butterfly (see Table 1) is $\Omega(d2^d)$ away from the lower bound we proved in Section 2 (see Corollary 4). In what follows we prove that the algorithm presented in [9] actually finds an asymptotically optimal feedback vertex set.

Recall that a d–dimensional butterfly is a graph $B_d = (V, E)$ composed by $(d+1)2^d$ vertices organized in $d+1$ levels of 2^d vertices each, where $v_{i,j}$ denotes the j–th vertex at level i, with $0 \leq i \leq d$ and $0 \leq j \leq 2^d - 1$. For $i > 0$, $v_{i,j}$ is connected with the two vertices $v_{i-1,j}$ and v_{i-1,j_i}, where j_i denotes the integer whose binary representation differs from that of j in only the i–th bit position.

Algorithm BUTTERFLY-FVS
  $S := \emptyset$
  for $i := 1$ to d do
    for $j := 0$ to $2^d - 1$ do
      if $(v_{i-1,j} \notin S)$ and $(v_{i,i_j} \notin S)$ then $S := S \cup \{v_{i,j}\}$

Luccio in [9] proves that after the execution of algorithm BUTTERFLY-FVS on a butterfly B_d, S is a feedback vertex set of B_d. The execution of the algorithm BUTTERFLY–FVS on a 4–dimensional butterfly is depicted in Fig. 5.

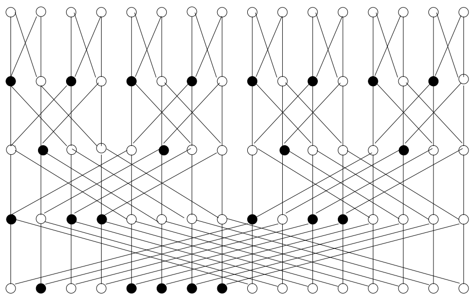

Fig. 4. The execution of the algorithm on a 4–dimensional butterfly. Black vertices are the vertices of the feedback vertex set S.

We denote by $b_i(d)$ the number of vertices of level i which belong to S and by F_d the size of S. Clearly, $b_0(d) = 0$, and since there are 2^d nodes at each level

$$b_i(d) = \frac{2^d - b_{i-1}(d)}{2},\ 1 \leq i \leq d. \tag{6}$$

Using (6) and the fact that $b_0(d) = 0$ and $b_d(d) \leq 2^{d-1}$, we obtain that

$$\begin{aligned}
F_d &= \sum_{i=0}^{d} b_i(d) \\
&= \sum_{i=1}^{d} \frac{2^d - b_{i-1}(d)}{2} \\
&= d2^{d-1} - \frac{\sum_{i=0}^{d-1} b_i(d)}{2} \\
&= d2^{d-1} - \frac{\sum_{i=0}^{d} b_i(d)}{2} + \frac{b_d(d)}{2} \\
&\leq d2^{d-1} - \frac{F_d}{2} + 2^{d-2} \Rightarrow \\
F_d &\leq \frac{(d+1/2)2^d}{3}
\end{aligned}$$

Thus, we have proved the following

Theorem 9 *The minimum feedback vertex set of the d–dimensional butterfly has size at most* $\frac{(d+1/2)2^d}{3}$.

References

[1] V. Bafna, P. Berman, and T. Fujito. A 2–Approximation Algorithm for the Undirected Feedback Vertex Set Problem. *SIAM Journal on Discrete Mathematics*, Vol. 12, n. 3, pp. 289–297, 1999.

[2] R. Bar–Yehuda, D. Geiger, J. Naor, and R. M. Roth. Approximation Algorithms for the Feedback Vertex Set Problem with Applications to Constraint Satisfaction and Bayesian Inference. *SIAM Journal on Computing*, Vol. 27(4), pp. 942–959, 1998.

[3] R. Focardi, F. L. Luccio, and D. Peleg. Feedback Vertex Set in Hypercubes. *Information Processing Letters*, Vol. 76, pp. 1–5, 2000.

[4] M. R. Garey and D. S. Johnson. Computers and Intractability. Freeman, San Francisco, CA, 1979.

[5] J. Kleinberg and A. Kumar. Wavelength Conversion in Optical Networks. In *Proc. of the 10th Annual ACM–SIAM Symposium on Discrete Algorithms (SODA 99)*, pp. 566–575, 1999.

[6] F.T. Leighton. Introduction to parallel algorithms and architectures. *Morgan Kaufmann Publishers*, 1992.

[7] Y. D. Liang and M. S. Chang. Minimum Feedback Vertex Set in Comparability Graphs and Convex Bipartite Graphs. *Acta Informatica*, Vol. 34, pp. 337–346, 1997.

[8] C. Lu and C. Tang. A Linear–Time Algorithm for the Weighted Feedback Vertex Problem on Interval Graphs. *Information Processing Letters*, Vol. 61(2), pp. 107–112, 1997.

[9] F. L. Luccio. Almost exact minimum feedback vertex set in meshes and butterflies. *Information Processing Letters*, Vol. 66, pp. 59-64, 1998.

[10] D. Peleg. Local Majority Voting, Small Coalitions and Controlling Monopolies in Graphs: A Review. In *Proc. of the 3rd Colloquium on Structural Information and Communication Complexity (SIROCCO 96)*, pp. 152–169, 1996.

[11] A. Shamir. A Linear Time Algorithm for Finding Minimum Cutsets in Reducible Graphs. *SIAM Journal on Computing*, Vol. 8(4), pp. 645–655, 1979.

[12] O. Togni. Placement de convertisseurs de longueurs d' ondes dans les réseaux optiques. In *Proc. of ALGOTEL 2000*, 2000.

[13] C. Wang, E. L. Lloyd, M. L. Sofra. Feedback Vertex Sets and Cyclically Reducible Graphs. *Journal of the ACM*, Vol. 32(2), pp. 296–313, 1985.