Convexity check

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1 Preliminaries

Recall Expression (22) in our current paper:

$$\frac{\partial \ln F_m}{\partial m_{i,r}^{j,t}} = \frac{1}{2} \ln m_{i,r}^{j,t} - \ln \mu_{i,r}^{j,t} + \frac{3}{10} \tag{1}$$

Taking advantage:

$$m_{i,r}^{j,t} = \mu_{i,r}^{j,t} \frac{\partial \Phi(\widetilde{\mu})}{\partial \mu_{i,r}^{j,t}} \Leftrightarrow m_{i,r}^{j,t} = \frac{\partial \Phi(\widetilde{\mu})}{\partial \ln \mu_{i,r}^{j,t}} \tag{2}$$

we can get:

$$\nabla^2_{m_{i,r}^{j,t}} \ln F_m \quad = \quad \left[\frac{1}{2m_{i,-1}^{j,-1}}, \frac{1}{2m_{i,-1}^{j,1}}, \frac{1}{2m_{i,1}^{j,-1}}, \frac{1}{2m_{i,1}^{j,1}} \right] \times \mathrm{I}_{4,4} - \left(\nabla^2_{\ln \mu_{i,r}^{j,t}} \Phi(\widetilde{\mu}) \right)^{-1} \tag{3}$$

We wish to show that the LPDMs of (3) alternate properly in sing, for each 4tuple of $m > 0$'s (which implicitly determines the 4tuple of μ's) that sum to a fixed positive constant $n_i^j > 0$, i.e. under the constraints:

$$\begin{aligned} m_{i,r}^{j,t} + m_{i,r}^{j,-t} &\leq 4 \left(m_{i,-r}^{j,t} + m_{i,-r}^{j,-t} \right) \\ m_{i,r}^{j,t} + m_{i,-r}^{j,t} &\leq 4 \left(m_{i,r}^{j,-t} + m_{i,-r}^{j,-t} \right) \\ m_{i,r}^{j,t} &= m_{i+r,-r}^{j+t,-t} \\ \sum_{r,t} m_{i,r}^{j,t} &= n_i^j \end{aligned} \tag{4}$$

A nice observation is that we can take advantage of (2) and transform (3) to (5), as a function of μ's only. Then proper alternation of signs of (5), for *all positive* μ's, will imply that the LPDMs of (3) alternate properly in sing for each 4tuple of m's that satisfy constraints (4).

$$\nabla^2_{m_{i,r}^{j,t}} \ln F_m \quad = \quad \left[\frac{1}{2\mu_{i,-1}^{j,-1} \frac{\partial \Phi(\widetilde{\mu})}{\partial \mu_{i,-1}^{j,-1}}}, \frac{1}{2\mu_{i,-1}^{j,1} \frac{\partial \Phi(\widetilde{\mu})}{\partial \mu_{i,-1}^{j,1}}}, \frac{1}{2\mu_{i,1}^{j,-1} \frac{\partial \Phi(\widetilde{\mu})}{\partial \mu_{i,1}^{j,-1}}} \frac{1}{2\mu_{i,1}^{j,1} \frac{\partial \Phi(\widetilde{\mu})}{\partial \mu_{i,1}^{j,1}}} \right] \times \mathrm{I}_{4,4} - \left(\nabla^2_{\ln \mu_{i,r}^{j,t}} \Phi(\widetilde{\mu}) \right)^{-1} \tag{5}$$

Observe that each entry in (5) is a *regular* polynomial, i.e. it consists of monomials that their exponents sum to a fixed constant. Then to show proper alternation of signs of (5) for *all positive* μ's, it suffices to show it for all 4tuples of μ's that sum *only* to 1. For example, let: $f(x,y,z,w) = x^2y^2zw^3 - 10xyz^3w^5 + x^5w^3$ with exponents summing to 8. Then,

$$\begin{aligned} f(x,y,z,w) &= (x^2y^2zw^3 - 10xyz^3w^5 + x^5w^3)\frac{(x+y+z+w)^8}{(x+y+z+w)^8} \\ &= \left(p_x^2p_y^2p_zp_w^3 - 10p_xp_yp_z^3p_w^5 + p_x^5p_w^3\right)(x+y+z+w)^8 \\ &\quad \text{with: } (x+y+z+w)^8 > 0 \end{aligned} \tag{6}$$

where $p_t = \frac{t}{x+y+z+w} \in (0,1]$ for each subscript $t \in \{x,y,z,w\}$. In other words, the sign of (6) depends *only* on the scaled vector $(p_x,\ldots,p_w) \in (0,1]^4$ of the positive variables x,y,z,w.

The C program: We represent the 4 μ's as $\mu 1,\ldots,\mu 4$ and the 4 m's as $m1,\ldots,m4$. Then $\mu 1$ ranges in $[bound, 1-bound]$ with `step`$= 1/5000$ and `bound`$= 0.00000001$. For each fixed value of $\mu 1$ then $\mu 2$ ranges in $[bound, 1-\mu 1]$ with `step`$= 1/5000$. For each fixed tuple of values $(\mu 1, \mu 2)$ then $\mu 3$ ranges in $[bound, 1-\mu 1-\mu 2]$ with `step`$= 1/5000$. Finally, for each tuple of values $(\mu 1, \mu 2, \mu 3)$ then we set $\mu 4 = 1-\mu 1-\mu 2-\mu 3 \geq$ `bound`.

The 2 matrices in Expression (5) are implemented as follows. The leftmost matrix

$$\left[\frac{1}{2\mu 1\frac{\partial\Phi(\widetilde{\mu})}{\partial\mu 1}}, \frac{1}{2\mu 2\frac{\partial\Phi(\widetilde{\mu})}{\partial\mu 2}}, \frac{1}{2\mu 3\frac{\partial\Phi(\widetilde{\mu})}{\partial\mu 3}}\,\frac{1}{2\mu 4\frac{\partial\Phi(\widetilde{\mu})}{\partial\mu 4}}\right] \times \mathrm{I}_{4,4}$$

in (5) is evaluated per tuple of mu's in the C program in the 4×4 matrix `Minvf`. The rightmost matrix

$$\nabla^2_{\ln\mu 1,\ldots,\ln\mu 4}\Phi(\widetilde{\mu})$$

in (5) is evaluated per tuple of mu's in the C program as the 4×4 matrix `Sf`, while its corresponding inverse in 4×4 matrix `Sinvf`. Since μ's add to 1, it is convenient to use the 3×3 matrix

$$HG = A \times (Minvf - Sinf) \times transpose(A)$$

where:

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Finally, the LPDMs of matrix `HG` are denoted as `currentLPDM1`, …, `currentLPDM3`.