The Impact of Altruism on the Efficiency of Atomic Congestion Games*

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Abstract. We study the effect of combining selfishness and altruism in atomic congestion games. We allow players to be partially altruistic and partially selfish and determine the impact of this behavior on the overall system performance. Surprisingly, our results indicate that, in general, by allowing players to be (even partially) altruistic, the overall system performance deteriorates. Instead, for the class of symmetric load balancing games, a balance between selfish and altruistic behavior improves system performance to optimality.

1 Introduction

Congestion games provide a natural model for antagonistic resource allocation in large-scale systems and have recently played a central role in algorithmic game theory. In a congestion game, a set of non-cooperative players, each controlling an unsplittable unit demand, compete over a set of resources. All players using a resource experience a latency (or cost) given by a non-negative and non-decreasing function of the total demand (or congestion) of the resource. Among a given set of resource subsets (or strategies), each player selects one selfishly trying to minimize her individual total cost, i.e., the sum of the latencies on the resources in the chosen strategy. Load balancing games are congestion games in which the strategies of the players are singletons. Load balancing games in which all players have all resources as singleton strategies are called symmetric.

A typical example of a congestion game stems from antagonistic routing on a communication network. In this setting, we have several network users, where each user wishes to send traffic between a source-destination pair of network nodes. Each user may select among all possible paths connecting her source-destination pair of nodes. A natural objective for a user is to route her traffic using as less congested links as possible. This situation can be modelled by a congestion game where the users of the network are the players and the communication links correspond to the resources. In a load balancing game, we may

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think of the resources as servers and the players as clients wishing to get served by one of the servers. Then, the load balancing game is used to model the inherent selfishness of the clients in the sense that each of them desires to be served by the least loaded server.

A natural solution concept that captures stable outcomes in a (congestion) game is that of a pure Nash equilibrium (PNE), a configuration where no player can decrease her individual cost by unilaterally changing her strategy. Rosenthal [13] proved that the PNE of congestion games correspond to the local optima of a natural potential function, and thus every congestion game admits a PNE. Much of the recent literature on congestion games has focused on quantifying the inefficiency due to the players' selfish behavior. It is well known that a PNE may not optimize the system performance, usually measured by the total cost incurred by all players. The main tool for quantifying and understanding the performance degradation due to selfishness has been the *price of anarchy*, introduced by Koutsoupias and Papadimitriou [10] (see also [12]). The price of anarchy is the worst-case ratio of the total cost of a PNE to the optimal total cost.

Many recent papers have provided tight upper and lower bounds on the price of anarchy for several interesting classes of congestion games, mostly congestion games with linear and polynomial latencies. Awerbuch et al. [2] and Christodoulou and Koutsoupias [7] proved that the price of anarchy of congestion games is 5/2 for linear latencies and $d^{\Theta(d)}$ for polynomial latencies of degree d. Subsequently, Aland et al. [1] obtained exact bounds on the price of anarchy for congestion games with polynomial latencies. Caragiannis et al. [4] proved that the same bounds hold for load balancing games as well. For symmetric load balancing games, Lücking et al. [11] proved that the price of anarchy is 4/3.

In this paper, we are interested in the impact of altruistic behavior on the efficiency of atomic congestion games with linear latency functions. We assume that a player with completely altruistic behavior aims to minimize the total latency incurred by the other players. We also consider types of behavior that lie between completely altruistic behavior and selfishness. In this respect, we use a parameter $\xi \in [0, 1]$ and consider a player to be ξ-altruistic is she aims to minimize the linear combination of the total latency incurred by the other players and her latency with coefficients ξ and $1 - \xi$ respectively. Hence, an 1-altruistic player acts completely altruistically while a 0-altruistic one is selfish.

Intuitively, altruism should be considered as a synonym for trustworthy behavior. In contrast to this intuition, we demonstrate rather surprising results. We show that having players that behave completely altruistically may lead to a significant deterioration of performance. More importantly, even a small degree of altruism may have a negative effect on performance compared to the case of selfish players. These results hold for general atomic congestion games in which players may have different strategy sets. This asymmetry seems to be incompatible with altruism. On the contrary, in simpler games such as symmetric load balancing games, we prove that a balance between altruism and selfishness in the players' behavior leads to optimal performance.

In technical terms, we show the following results which extend the known bounds on the price of anarchy of games with selfish players to games with ξ-altruistic ones:

- The price of anarchy of atomic congestion games with ξ-altruistic players is at most $\frac{5-\xi}{2-\xi}$ when $\xi \in [0, 1/2]$ and at most $\frac{2-\xi}{1-\xi}$ when $\xi \in [1/2, 1]$. These bounds are proved to be tight for all values of ξ. The corresponding lower bound proofs are based on the construction of load balancing games with the desired price of anarchy.
- For symmetric load balancing games, we show that the price of anarchy with ξ-altruistic players is at most $\frac{4(1-\xi)}{3-2\xi}$ when $\xi \in [0, 1/2]$ and at most $\frac{3-2\xi}{4(1-\xi)}$ when $\xi \in [1/2, 1]$. These bounds are proved to be tight as well; the lower bound constructions are very simple and use symmetric load balancing games with two machines and two players.

Surprisingly, our first set of results indicates that altruism may be harmful in general since the price of anarchy increases from 5/2 to unbounded as the degree of altruism increases from 0 to 1. Hence, selfishness is more beneficial than altruism in general. Our second set of results establishes a different setting for symmetric load balancing games. Interestingly, a balance between altruistic and selfish behavior leads to optimal performance (i.e., the price of anarchy is 1 and the equilibria reached are optimal). This has to be compared to the tight bound of 4/3 on the price of anarchy with selfish players. Again, completely altruistic behavior leads to an unbounded price of anarchy.

In our upper bound proofs, we follow the standard high-level analysis ideas that have been used in the literature (see [3]) in order to compare the cost of equilibria to the cost of optimal assignments but adapt it to the case of altruistic players. For each player, we express with an inequality its preference to the strategy she uses in the equilibrium instead of the one she uses in the optimal assignment. For general atomic congestion games, by summing these inequalities over all players, we obtain an upper bound on the cost of the equilibrium in terms of quantities characterizing both the equilibrium and the optimal assignment. Then, we need to use new inequalities on the non-negative integers in order to obtain a direct relation between the cost of the equilibrium and the optimal assignment. In symmetric load balancing, we exploit the symmetry in order to obtain a better relation between the cost of the equilibrium and the optimal cost. In our analysis, we use the inequalities expressing the preference of a carefully selected set of players and develop new inequalities over non-negative integers in order to obtain our upper bound.

Chen and Kempe [6] have considered similar questions in non-atomic congestion games, i.e., games with an infinite number of players each controlling a negligibly small amount of traffic. Our findings are inherently different than theirs as in non-atomic congestion games the system performance improves as the degree of altruism of the players increases. Hoefer and Skopalik [9] consider atomic congestion games using a slightly different definition of altruism, which corresponds to ξ-altruistic behavior with $\xi \in [0, 1/2]$ in our model. They mainly present complexity results for the computation of equilibria in the corresponding congestion games and do not address questions related to the price of anarchy.

The rest of the paper is structured as follows. We begin with preliminary definitions and properties of altruistic players in Section 2. Our upper bounds for atomic congestion games and the corresponding lower bounds are presented in Sections 3 and 4, respectively. Section 5 is devoted to our results regarding symmetric load balancing. We conclude in Section 6 with a discussion on possible extensions of our work.

2 Preliminaries

In this section we formally define the model and establish characteristic inequalities that capture the players' behavior.

In *atomic congestion games* there is a set E of resources, each resource e having a non-negative and non-decreasing latency function f_e defined over non-negative numbers, and a set of n players. Each player i has a set of strategies $S_i \subseteq 2^E$ (each strategy of player i is a set of resources) and controls an unsplittable unit demand. An assignment $A = (A_1, ..., A_n)$ is a vector of strategies, one strategy for each player. The cost of player i for an assignment A is defined as $cost_i(A) = \sum_{e \in A_i} f_e(n_e(A))$, where $n_e(A)$ is the number of players using resource e in A, while the *social cost* of an assignment is the total cost of all players. An assignment is a *pure Nash equilibrium* if no player has an incentive to unilaterally deviate to another strategy, i.e., $cost_i(A) \leq cost_i(A_{-i}, s)$ for any player i and for any $s \in S_i$, where (A_{-i}, s) is the assignment produced from A if player i deviates from A_i to s. This inequality is also known as the *Nash condition*. A congestion game is called *symmetric* when all players share the same set of strategies. Load balancing games are congestion games where the strategies of the players are singleton sets. The *price of anarchy* of a congestion game is defined as the ratio of the maximum social cost over all Nash equilibria over the optimal cost. The price of anarchy for a class of congestion games is simply the highest price of anarchy among all games belonging to that class.

In this paper, we consider latency functions of the form $f_e(x) = \alpha_e x + \beta_e$ for each resource e, where α_e, β_e are non-negative constants. Then, the cost of a player i for an assignment A becomes $cost_i(A) = \sum_{e \in A_i} (\alpha_e n_e(A) + \beta_e)$, while the social cost becomes

$$\sum_i cost_i(A) = \sum_i \sum_{e \in A_i} (\alpha_e n_e(A) + \beta_e) = \sum_e \left(\alpha_e n_e^2(A) + \beta_e n_e(A)\right).$$

We now proceed to modify the model so that altruism is taken into account. We assume that each player i is partially altruistic, in the sense that she tries to minimize a function depending on the total cost of all other players and the total latency she experiences. We say that player i following a strategy A_i is *ξ-altruistic*, where $\xi \in [0, 1]$, when her cost function is

$$cost_i(A) = \xi \left(\sum_e \left(\alpha_e n_e^2(A) + \beta_e n_e(A)\right) - \sum_{e \in A_i} (\alpha_e n_e(A) + \beta_e) \right)$$

$$+(1-\xi)\sum_{e\in A_i}(\alpha_e n_e(A)+\beta_e).$$

Clearly, when $\xi = 0$ then player i wishes to minimize her total latency, while when $\xi = 1$ player i wishes to minimize the total latency of all other players.

Now, consider two assignments A and A' that differ in the strategy of player i and let p_1 and p_2 be the strategies of i in the two assignments. Furthermore, by slightly abusing notation, we let $n_e = n_e(A)$ and $n'_e = n_e(A')$.

Assume that assignment A is an equilibrium; the cost of player i under A is

$$\begin{aligned}
cost_i(A) &= \xi\left(\sum_e\left(\alpha_e n_e^2+\beta_e n_e\right)-\sum_{e\in p_1}(\alpha_e n_e+\beta_e)\right)+(1-\xi)\sum_{e\in p_1}(\alpha_e n_e+\beta_e)\\
&= \xi\left(\sum_{e\notin p_1\ominus p_2}\left(\alpha_e n_e^2+\beta_e n_e\right)+\sum_{e\in p_1\ominus p_2}\left(\alpha_e n_e^2+\beta_e n_e\right)\right)\\
&\quad +(1-2\xi)\left(\sum_{e\in p_1\cap p_2}(\alpha_e n_e+\beta_e)+\sum_{e\in p_1\setminus p_2}(\alpha_e n_e+\beta_e)\right),
\end{aligned}$$

where $\ominus$ is the symmetric difference operator in set theory, i.e., for two sets a, b it holds that $a\ominus b=(a\setminus b)\cup(b\setminus a)$.

Consider now the second assignment $A' = (A_{-i}, p_2)$ in which player i has changed her strategy from p_1 to p_2. Observe that $n'_e = n_e + 1$ for $e \in p_2 \setminus p_1$, $n'_e = n_e - 1$ for $e \in p_1 \setminus p_2$ and $n'_e = n_e$ otherwise. Her cost under the second assignment is

$$\begin{aligned}
cost_i(A') &= \xi\left(\sum_e\left(\alpha_e n_e'^2+\beta_e n'_e\right)-\sum_{e\in p_2}(\alpha_e n'_e+\beta_e)\right)+(1-\xi)\sum_{e\in p_2}(\alpha_e n'_e+\beta_e)\\
&= \xi\left(\sum_{e\notin p_1\ominus p_2}\left(\alpha_e n_e'^2+\beta_e n'_e\right)+\sum_{e\in p_1\ominus p_2}\left(\alpha_e n_e'^2+\beta_e n'_e\right)\right)\\
&\quad +(1-2\xi)\left(\sum_{e\in p_1\cap p_2}(\alpha_e n'_e+\beta_e)+\sum_{e\in p_2\setminus p_1}(\alpha_e n'_e+\beta_e)\right)\\
&= \xi\left(\sum_{e\notin p_1\ominus p_2}\left(\alpha_e n_e^2+\beta_e n_e\right)+\sum_{e\in p_1\setminus p_2}\left(\alpha_e\left(n_e-1\right)^2+\beta_e\left(n_e-1\right)\right)\right.\\
&\quad \left.+\sum_{e\in p_2\setminus p_1}\left(\alpha_e\left(n_e+1\right)^2+\beta_e\left(n_e+1\right)\right)\right)\\
&\quad +(1-2\xi)\left(\sum_{e\in p_1\cap p_2}(\alpha_e n_e+\beta_e)+\sum_{e\in p_2\setminus p_1}(\alpha_e\left(n_e+1\right)+\beta_e)\right).
\end{aligned}$$

Since player i has no incentive to change her strategy from p_1 to p_2, we obtain that $cost_i(A) \leq cost_i(A')$, i.e.,

$$\xi \sum_{e \in p_1 \ominus p_2} \left(\alpha_e n_e^2 + \beta_e n_e\right) + (1-2\xi) \sum_{e \in p_1 \setminus p_2} (\alpha_e n_e + \beta_e) \leq$$
$$\xi \left(\sum_{e \in p_1 \setminus p_2} \left(\alpha_e (n_e - 1)^2 + \beta_e (n_e - 1)\right) + \sum_{e \in p_2 \setminus p_1} \left(\alpha_e (n_e + 1)^2 + \beta_e (n_e + 1)\right) \right)$$
$$+ (1-2\xi) \sum_{e \in p_2 \setminus p_1} (\alpha_e (n_e + 1) + \beta_e),$$

which implies that

$$\begin{aligned}\sum_{e \in p_1 \setminus p_2} (\alpha_e (n_e - \xi) + \beta_e (1-\xi)) &\leq \sum_{e \in p_2 \setminus p_1} (\alpha_e (n_e + 1 - \xi) + \beta_e (1-\xi)) \\ &= \sum_{e \in p_2 \setminus p_1} (\alpha_e (n_e' - \xi) + \beta_e (1-\xi)),\end{aligned}$$

and, equivalently,

$$\sum_{e \in p_1} (\alpha_e (n_e - \xi) + \beta_e (1-\xi)) \leq \sum_{e \in p_2} (\alpha_e (n_e' - \xi) + \beta_e (1-\xi)).$$

Observe that when $\xi = 0$, the above inequality is merely the Nash condition. In general, this condition implies that, given an assignment A_{-i} of the remaining players, a ξ-altruistic player i aims to select a strategy s from S_i such that the expression

$$\sum_{e \in s} (\alpha_e (n_e(A_{-i}, s) - \xi) + \beta_e (1-\xi))$$

is minimized.

In the rest of this paper, we will assume, without loss of generality, that $\beta_e = 0$ for all resources. Our lower bound constructions exhibit this property, while the proofs of our upper bounds carry over even with non-zero values of β_e.

3 Upper Bounds for Atomic Congestion Games

In this section we describe our upper bounds concerning the price of anarchy for atomic congestion games and ξ-altruistic players. In our proofs we use the following two technical lemmas.

Lemma 1. *For all integers $x, y \geq 0$ and $\xi \in [0, 1/2]$ it holds that*

$$xy + (1-\xi)\, y + \xi x \leq \frac{1+\xi}{3} x^2 + \frac{5-\xi}{3} y^2.$$

Proof. Consider the function

$$f(x,y) = \frac{1+\xi}{3}x^2 + \frac{5-\xi}{3}y^2 - xy - (1-\xi)\,y - \xi x.$$

It suffices to prove that $f(x,y) \geq 0$ when x, y are non-negative integers and $\xi \in [0, 1/2]$.

We start with the case $x = y = k$. Then,

$$f(x,y) = f(k,k) = k^2 - k \geq 0.$$

We now consider the case $x = k$ and $y = k + z$, where $k \geq 0$ and $z \geq 1$. Then,

$$\begin{aligned} f(x,y) &= f(k, k+z) \\ &= f(k,k) + \frac{5-\xi}{3}\left(z^2 + 2zk\right) - kz - (1-\xi)\,z \\ &= f(k,k) + z\left(\frac{5-\xi}{3}z + \frac{7-2\xi}{3}k - 1 + \xi\right). \end{aligned}$$

Since $f(k,k) \geq 0$, $z \geq 1$ and $\xi \in [0, 1/2]$, we conclude that $f(x,y) \geq 0$, when $y > x$.

Finally, we consider the case where $x = k + z$ and $y = k$, where $k \geq 0$ and $z \geq 1$. Then,

$$\begin{aligned} f(x,y) &= f(k+z, k) \\ &= \frac{1+\xi}{3}(k+z)^2 + \frac{5-\xi}{3}k^2 - (k+z)\,k - (1-\xi)\,k - \xi\,(k+z) \\ &= k^2 - k + z\left(\frac{1+\xi}{3}(z + 2k) - k - \xi\right). \end{aligned}$$

If $z > k$, then

$$f(x,y) \geq k^2 - k + z\left((1+\xi)\,k + \frac{1+\xi}{3} - k - \xi\right) \geq 0,$$

since $k \geq 0$ and $\xi \in [0, 1/2]$.

If $z = k$, then

$$f(x,y) = k^2 - k + k\left((1+\xi)\,k - k - \xi\right) \geq 0,$$

since $k = z \geq 1$ and $\xi \in [0, 1/2]$.

Finally, if $z < k$, then

$$f(x,y) = k^2 - k + z\left(\frac{1+\xi}{3}z - \xi\right) + z\left(\frac{1+\xi}{3}2k - k\right).$$

Since $z\left(\frac{1+\xi}{3}z - \xi\right) \geq 0$ for $z \geq 1$ and $\xi \in [0, 1/2]$, and $k^2 - k - zk \geq 0$ for $z \leq k - 1$, the lemma follows. □

Lemma 2. *For all integers $x, y \geq 0$ and $\xi \in [1/2, 1]$ it holds that*

$$xy + (1-\xi)\,y + \xi x \leq \xi x^2 + (2-\xi)\,y^2.$$

Proof. Consider the function

$$f(x,y) = \xi x^2 + (2-\xi)\,y^2 - xy - (1-\xi)\,y - \xi x.$$

To prove the lemma it suffices to show that $f(x,y) \geq 0$ when x, y are non-negative integers and $\xi \in [1/2, 1]$.

We first consider the case where $x = y = k$. Then,

$$f(x,y) = f(k,k) = k^2 - k \geq 0.$$

We now consider the case $x > y$ and let $x = k + z$ and $y = k$, where $k \geq 0$ and $z \geq 1$. Then,

$$\begin{aligned} f(x,y) &= f(k+z,k) \\ &= f(k,k) + \xi\left(z^2 + 2kz\right) - kz - \xi z \\ &= f(k,k) + z\,(z\xi + 2\xi k - k - \xi) \\ &= f(k,k) + z\,(\xi\,(z-1) + k\,(2\xi - 1))\,. \end{aligned}$$

Since $f(k,k) \geq 0$, $z \geq 1$ and $\xi \in [1/2, 1]$, it holds that $f(x,y) \geq 0$ when $x > y$.

Finally, we consider the case $y > x$ and let $x = k$ and $y = k + z$, where $k \geq 0$ and $z \geq 1$. Then,

$$\begin{aligned} f(x,y) &= f(k,k+z) \\ &= f(k,k) + (2-\xi)\left(z^2 + 2kz\right) - kz - (1-\xi)z \\ &= f(k,k) + z\,((2-\xi)\,(z+2k) - k - 1 + \xi) \\ &= f(k,k) + z\,((2-\xi)\,z + (3-2\xi)\,k - 1 + \xi)\,. \end{aligned}$$

Since $f(k,k) \geq 0$, $z \geq 1$ and $\xi \in [1/2, 1]$, it holds that $f(x,y) \geq 0$ also when $y > x$. □

We note that the above lemmas also hold for the more general case of possibly negative x and y, but it suffices to consider non-negative values for our purposes. We are now ready to state the main result of this section.

Theorem 1. *The price of anarchy of atomic congestion games with ξ-altruistic players is at most $\frac{5-\xi}{2-\xi}$ if $\xi \in [0, 1/2]$ and at most $\frac{2-\xi}{1-\xi}$ if $\xi \in [1/2, 1]$.*

Proof. Consider a pure Nash equilibrium and an optimal assignment, and denote by n_e and o_e the number of players using resource e in the two assignments. Furthermore, let p_{i_1} and p_{i_2} be the strategies of player i in the two assignments. Since player i is a ξ-altruistic player, it holds that

$$\sum_{e \in p_{i_1}} \alpha_e\,(n_e - \xi) \leq \sum_{e \in p_{i_2}} \alpha_e\,(n_e + 1 - \xi).$$

For the total latency of the pure Nash equilibrium, it holds that

$$
\begin{aligned}
cost &= \sum_e \alpha_e n_e^2 = \sum_i \sum_{e \in p_{i_1}} \alpha_e n_e \\
&= \sum_i \sum_{e \in p_{i_1}} \left(\alpha_e \left(n_e - \xi\right) + \alpha_e \xi\right) \\
&\leq \sum_i \sum_{e \in p_{i_2}} \alpha_e \left(n_e + 1 - \xi\right) + \sum_i \sum_{e \in p_{i_1}} \alpha_e \xi \\
&= \sum_e \alpha_e n_e o_e + (1-\xi) \sum_e \alpha_e o_e + \xi \sum_e \alpha_e n_e \\
&= \sum_e \alpha_e \left(n_e o_e + (1-\xi)\, o_e + \xi n_e\right).
\end{aligned}
$$

So, for the case where $\xi \in [0, 1/2]$, from Lemma 1 we obtain that

$$\sum_e \alpha_e \left(n_e o_e + (1-\xi)\, o_e + \xi n_e\right) \leq \frac{1+\xi}{3} \sum_e \alpha_e n_e^2 + \frac{5-\xi}{3} \sum_e \alpha_e o_e^2,$$

and, thus,

$$\sum_e \alpha_e n_e^2 \leq \frac{1+\xi}{3} \sum_e \alpha_e n_e^2 + \frac{5-\xi}{3} \sum_e \alpha_e o_e^2$$

which leads to

$$\frac{2-\xi}{3} \sum_e \alpha_e n_e^2 \leq \frac{5-\xi}{3} \sum_e \alpha_e o_e^2.$$

So, we obtain that the price of anarchy for this case is

$$\frac{\sum_e \alpha_e n_e^2}{\sum_e \alpha_e o_e^2} \leq \frac{5-\xi}{2-\xi}.$$

Similarly, for the case where $\xi \in [1/2, 1]$, from Lemma 2 we obtain that

$$\sum_e \alpha_e \left(n_e o_e + (1-\xi)\, o_e + \xi n_e\right) \leq \xi \sum_e \alpha_e n_e^2 + (2-\xi) \sum_e \alpha_e o_e^2,$$

and, thus,

$$\sum_e \alpha_e n_e^2 \leq \xi \sum_e \alpha_e n_e^2 + (2-\xi) \sum_e \alpha_e o_e^2$$

which leads to

$$(1-\xi) \sum_e \alpha_e n_e^2 \leq (2-\xi) \sum_e \alpha_e o_e^2.$$

So, we obtain that the price of anarchy for this case is

$$\frac{\sum_e \alpha_e n_e^2}{\sum_e \alpha_e o_e^2} \leq \frac{2-\xi}{1-\xi}.$$

□

We observe that altruism is actually harmful, since the price of anarchy is minimized when $\xi = 0$, i.e., in the absence of altruism. Furthermore, when $\xi = 1$, i.e., players are completely altruistic, the price of anarchy is unbounded.

4 Lower Bounds for Atomic Congestion Games

In this section we state our lower bounds on the price of anarchy. The constructions in the proofs are load balancing games and are similar to a construction used in [4]. In these constructions, we represent the load balancing game as a graph. In this graph, each node represents a machine, and each edge represents a player having as possible strategies the machines corresponding to the nodes defining the edge.

Theorem 2. *For any $\epsilon > 0$ and $\xi \in [0, 1/2]$, there is a load balancing game with ξ-altruistic users whose price of anarchy is at least $\frac{5-\xi}{2-\xi} - \epsilon$.*

Proof. We construct a graph G, consisting of a complete binary tree with $k+1$ levels and $2^{k+1}-1$ nodes, with a line of $k+1$ edges and $k+1$ additional nodes hung at each leaf. So, graph G has $2k+2$ levels $0, \ldots, 2k+1$, with 2^i nodes at level i for $i = 0, \ldots, k$ and 2^k nodes at levels $k+1, \ldots, 2k+1$. The machines corresponding to nodes of level $i = 0, \ldots, k-1$, have latency functions $f_i(x) = (\frac{2-\xi}{3-\xi})^i x$, the machines corresponding to nodes of level $i = k, \ldots, 2k$, have latency functions $f_i(x) = (\frac{2-\xi}{3-\xi})^{k-1}(\frac{1-\xi}{2-\xi})^{i-k}x$, and the machines corresponding to nodes of level $2k+1$, have latency functions $f_{2k+1}(x) = (\frac{2-\xi}{3-\xi})^{k-1}(\frac{1-\xi}{2-\xi})^k x$. Consider the assignment where all players select machines corresponding to the endpoint of their corresponding edge which is closer to the root of graph G. It is not hard to see that this is a Nash equilibrium, since machines corresponding to nodes of level $i = 0, \ldots, k-1$, have two players and latency $2(\frac{2-\xi}{3-\xi})^i$, machines corresponding to nodes of level $i = k, \ldots, 2k$, have one player and latency $(\frac{2-\xi}{3-\xi})^{k-1}(\frac{1-\xi}{2-\xi})^{i-k}$, and machines corresponding to nodes of level $2k+1$, have no player. Therefore, due to the definition of the latency functions, a player assigned to a machine corresponding to a node of level $i = 0, \ldots, 2k$, would experience exactly the same latency if she changed her decision and chose the machine corresponding to the node of level $i+1$. The cost of the assignment is

$$\begin{aligned}
cost &= \sum_{i=0}^{k-1} 4 \cdot 2^i \left(\frac{2-\xi}{3-\xi}\right)^i + \sum_{i=k}^{2k} 2^k \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{i-k} \\
&= 4\frac{\left(\frac{2(2-\xi)}{3-\xi}\right)^k - 1}{\frac{4-2\xi}{3-\xi} - 1} + 2^k \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1 - \left(\frac{1-\xi}{2-\xi}\right)^{k+1}}{1 - \frac{1-\xi}{2-\xi}}\right) \\
&= \frac{4(3-\xi)}{1-\xi}\left(\left(\frac{2(2-\xi)}{3-\xi}\right)^k - 1\right) + (2-\xi)\left(\frac{3-\xi}{2-\xi}\right)\left(\frac{2(2-\xi)}{3-\xi}\right)^k \\
&\quad -(2-\xi)2^k \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{k+1} \\
&= (3-\xi)\left(\frac{5-\xi}{1-\xi}\right)\left(\frac{2(2-\xi)}{3-\xi}\right)^k - \frac{(3-\xi)(1-\xi)}{2-\xi}\left(\frac{2-2\xi}{3-\xi}\right)^k - \frac{4(3-\xi)}{1-\xi}.
\end{aligned}$$

To compute the upper bound on the cost of the optimal assignment it suffices to consider the assignment where all players select the machines corresponding to nodes which are further from the root. We obtain that the cost *opt* of the optimal assignment is

$$\begin{aligned}
opt &\le \sum_{i=1}^{k-1} 2^i \left(\frac{2-\xi}{3-\xi}\right)^i + \sum_{i=k}^{2k} 2^k \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{i-k} \\
&+2^k \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{k} \\
&= \frac{3-\xi}{1-\xi}\left(\left(\frac{2(2-\xi)}{3-\xi}\right)^k - 1\right) - (2-\xi)2^k \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\left(\frac{1-\xi}{2-\xi}\right)^{k+1} - 1\right) \\
&+2^k \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{k} - 1 \\
&= \frac{3-\xi}{1-\xi}\left(\left(\frac{2(2-\xi)}{3-\xi}\right)^k - 1\right) - 1 + (2-\xi)\left(\frac{3-\xi}{2-\xi}\right)\left(\frac{2(2-\xi)}{3-\xi}\right)^k \\
&-2(2-\xi)\left(\frac{1-\xi}{2-\xi}\right)^{k+1}\left(\frac{2(2-\xi)}{3-\xi}\right)^{k-1} + 2^k \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{k} \\
&= \left(\frac{3-\xi}{1-\xi} + 3 - \xi\right)\left(\frac{2(2-\xi)}{3-\xi}\right)^k - 2(2-\xi)\left(\frac{1-\xi}{2-\xi}\right)^{k+1}\left(\frac{2(2-\xi)}{3-\xi}\right)^{k-1} \\
&+2^k \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{k} - \frac{3-\xi}{1-\xi} - 1.
\end{aligned}$$

Hence, for any $\epsilon > 0$ and for sufficiently large k, the price of anarchy of the game is larger than

$$\frac{cost}{opt} \ge \frac{\frac{(3-\xi)(5-\xi)}{1-\xi}}{\frac{(3-\xi)(2-\xi)}{1-\xi}} - \epsilon = \frac{5-\xi}{2-\xi} - \epsilon.$$

□

We notice that this lower bound is tight for $\xi \in [0, 1/2]$. In order to prove a tight lower bound for the case $\xi \in [1/2, 1]$, it suffices to focus on one line of $k+2$ nodes and $k+1$ edges hanging from the binary tree of the aforementioned graph (including the corresponding leaf).

Theorem 3. *For any $\epsilon > 0$ and $\xi \in [1/2, 1]$, there is a load balancing game with ξ-altruistic users, whose price of anarchy is at least $\frac{2-\xi}{1-\xi} - \epsilon$.*

Proof. Consider the construction used in the proof of the previous theorem. We remind that the machine located at the node of the $2k+1$ level, has latency function $f_{2k+1}(x) = (\frac{2-\xi}{3-\xi})^{k-1}(\frac{1-\xi}{2-\xi})^k x$, and the machines corresponding to nodes of levels $i = k, \ldots, 2k$ have latency functions $f_i(x) = (\frac{2-\xi}{3-\xi})^{k-1}(\frac{1-\xi}{2-\xi})^{i-k} x$. Similarly, the assignment, where all players select the machine corresponding to the node closer to the root, is a Nash equilibrium, whereas the players are optimally assigned to the machine corresponding to the node further from the root (considering the endpoints of the corresponding edge). Using similar analysis, we obtain that

$$\begin{aligned}
cost &= \sum_{i=k}^{2k} \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{i-k} \\
&= \sum_{i=0}^{k} \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{i} \\
&= \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \frac{(\frac{1-\xi}{2-\xi})^{k+1}-1}{\frac{1-\xi}{2-\xi}-1} = (2-\xi)\left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(1-\left(\frac{1-\xi}{2-\xi}\right)^{k+1}\right),
\end{aligned}$$

and

$$\begin{aligned}
opt &\leq \sum_{i=k+1}^{2k} \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{i-k} + \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{k} \\
&= \sum_{i=1}^{k} \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{i} + \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{k} \\
&= \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \frac{(\frac{1-\xi}{2-\xi})^{k+1}-\frac{1-\xi}{2-\xi}}{\frac{1-\xi}{2-\xi}-1} + \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{k} \\
&= (2-\xi)\left(\frac{2-\xi}{3-\xi}\right)^{k-1} \frac{1-\xi}{2-\xi}\left(1-\left(\frac{1-\xi}{2-\xi}\right)^{k}\right) + \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{k} \\
&= (1-\xi)\left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(1-\left(\frac{1-\xi}{2-\xi}\right)^{k}\right) + \left(\frac{2-\xi}{3-\xi}\right)^{k-1} \left(\frac{1-\xi}{2-\xi}\right)^{k}.
\end{aligned}$$

We conclude, that for any $\epsilon > 0$, and sufficiently large k, the price of anarchy of the game is larger than $\frac{2-\xi}{1-\xi} - \epsilon$. □

5 Symmetric Load Balancing Games

In this section, we consider the important class of symmetric load balancing games with ξ-altruistic players. In our proof, we make use of the following two technical lemmas.

Lemma 3. *For any integers $x, y \geq 0$ and any $\xi \in [0, 1/2]$ it holds that, when $x < y$,*

$$xy + (1-\xi)\,y - (1-\xi)\,x \leq \frac{1+2\xi}{4}x^2 + (1-\xi)\,y^2,$$

and, when $x \geq y$,

$$xy + \xi x - \xi y \leq \frac{1+2\xi}{4}x^2 + (1-\xi)\,y^2.$$

Proof. We begin with the case $x < y$. Consider the function

$$f(x,y) = \frac{1+2\xi}{4}x^2 + (1-\xi)\,y^2 - xy - (1-\xi)\,y + (1-\xi)\,x.$$

It suffices to show that $f(x,y) \geq 0$. Let $y = x + z$, where z is a positive integer. Then

$$\begin{aligned}
f(x,y) &= f(x,x+z) \\
&= \frac{1+2\xi}{4}x^2 + (1-\xi)\left(x^2+z^2+2xz\right) - x^2 - xz + (1-\xi)\,x - (1-\xi)\,x \\
&\quad - (1-\xi)\,z \\
&= \left(\frac{1+2\xi}{4} + 1 - \xi - 1\right)x^2 + (1-\xi)\,z^2 + (2-2\xi-1)\,xz - (1-\xi)\,z \\
&= \frac{1-2\xi}{4}x^2 + (1-\xi)\,z^2 + (1-2\xi)\,xz - (1-\xi)\,z \\
&\geq 0,
\end{aligned}$$

since $x \geq 0$, $z \geq 1$ and $\xi \in [0, 1/2]$.

We now consider the case $x \geq y$. Consider the function

$$g(x,y) = \frac{1+2\xi}{4}x^2 + (1-\xi)\,y^2 - xy - \xi x + \xi y.$$

In order to complete the proof, we have to show that $g(x,y) \geq 0$. Since $x \geq y$, let $x = y + z$, where z is a non-negative integer. Then,

$$\begin{aligned}
g(x,y) &= g(y+z,y) \\
&= \frac{1+2\xi}{4}\,(y+z)^2 + (1-\xi)\,y^2 - (y+z)\,y - \xi\,(y+z) + \xi y \\
&= \frac{1+2\xi}{4}y^2 + \frac{1+2\xi}{4}z^2 + \frac{1+2\xi}{2}yz - \xi y^2 - yz - \xi z \\
&= \frac{1-2\xi}{4}y^2 + \frac{1+2\xi}{4}z^2 - \frac{1-2\xi}{2}yz - \xi z \\
&= \frac{1}{4}(y^2+z^2-2yz) - \frac{\xi}{2}(y^2-z^2-2yz) - \xi z \\
&= \frac{1}{4}(y-z)^2 - \frac{\xi}{2}(y^2+z^2-2yz) + \xi z^2 - \xi z \\
&= \frac{1}{4}(y-z)^2 - \frac{\xi}{2}(y-z)^2 + \xi z(z-1) \\
&\geq 0
\end{aligned}$$

since $z \geq 0$ and $\xi \in [0, 1/2]$. ∎

Lemma 4. *For any integers $x, y \geq 0$ and any $\xi \in [1/2, 1]$ it holds that when $x < y$*

$$xy + (1-\xi)\,y - (1-\xi)\,x \leq \xi x^2 + \frac{3-2\xi}{4}y^2,$$

and when $x \geq y$

$$xy + \xi x - \xi y \leq \xi x^2 + \frac{3-2\xi}{4} y^2.$$

Proof. The proof follows from Lemma 3. Note that the two inequalities of Lemma 3 can be transformed to those of Lemma 4 by replacing ξ by $1-\xi$ and exchanging x and y. Furthermore, the two inequalities become identical when $x = y$. □

Again, these two lemmas also hold when x, y can be negative. We are now ready to prove the main result of this section.

Theorem 4. *The price of anarchy for symmetric load balancing games with ξ-altruistic players is $\frac{4(1-\xi)}{3-2\xi}$ when $\xi \in [0, 1/2]$ and $\frac{3-2\xi}{4(1-\xi)}$ when $\xi \in [1/2, 1]$.*

Proof. Consider a pure Nash equilibrium and an optimal assignment and let n_j and o_j be the number of players in machine j in the equilibrium and the optimal assignment, respectively. Consider the sets H and L of machines j such that $n_j > o_j$ and $n_j < o_j$, respectively. Denote by S the set of players consisting of $n_j - o_j$ players that are in machine $j \in H$ in equilibrium, for every machine $j \in H$. Observe that $\sum_{j \in H} (n_j - o_j) = \sum_{j \in L} (o_j - n_j)$. Hence, we can associate each player of S with a machine in L such that $o_j - n_j$ players of S are associated to each machine $j \in L$.

Consider a player in S that lies in machine $j \in H$ in equilibrium and let $j' \in L$ be the machine of L she is associated with. By the ξ-altruistic condition, we have that $\alpha_j(n_j - \xi) \leq \alpha_{j'}(n_{j'} - \xi + 1)$. By summing up the ξ-altruistic conditions for each player in S, we obtain that

$$\sum_{j:n_j>o_j} \alpha_j(n_j - \xi)(n_j - o_j) \leq \sum_{j:n_j<o_j} \alpha_j(n_j - \xi + 1)(o_j - n_j). \tag{1}$$

Now using (1), the fact that n_j and o_j are integers, and the definition of the latency functions, we obtain that

$$\begin{aligned}
\sum_j \alpha_j n_j^2 &= \sum_{j:n_j>o_j} \alpha_j n_j^2 + \sum_{j:n_j \leq o_j} \alpha_j n_j^2 \\
&= \sum_{j:n_j>o_j} \alpha_j \left(n_j - \xi\right) \left(n_j - o_j\right) + \sum_{j:n_j>o_j} \left(\alpha_j n_j o_j + \alpha_j \xi n_j - \alpha_j \xi o_j\right) \\
&\quad + \sum_{j:n_j \leq o_j} \alpha_j n_j^2 \\
&\leq \sum_{j:n_j<o_j} \alpha_j(n_j - \xi + 1)(o_j - n_j) + \sum_{j:n_j>o_j} \left(\alpha_j n_j o_j + \alpha_j \xi n_j - \alpha_j \xi o_j\right) \\
&\quad + \sum_{j:n_j \leq o_j} \alpha_j n_j^2 \\
&= \sum_{j:n_j<o_j} \left(\alpha_j n_j o_j - \alpha_j n_j^2 - \alpha_j \xi o_j + \alpha_j \xi n_j + \alpha_j o_j - \alpha_j n_j\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j:n_j>o_j} (\alpha_j n_j o_j + \alpha_j \xi n_j - \alpha_j \xi o_j) + \sum_{j:n_j \le o_j} \alpha_j n_j^2 \\
&= \sum_{j:n_j<o_j} \alpha_j \left(n_j o_j - \xi o_j + \xi n_j + o_j - n_j\right) + \sum_{j:n_j>o_j} \alpha_j \left(n_j o_j + \xi n_j - \xi o_j\right) \\
&+ \sum_{j:n_j=o_j} \alpha_j n_j^2 \\
&= \sum_{j:n_j<o_j} \alpha_j \left(n_j o_j + (1-\xi)\left(o_j - n_j\right)\right) + \sum_{j:n_j \ge o_j} \alpha_j \left(n_j o_j + \xi \left(n_j - o_j\right)\right).
\end{aligned}
$$

When $\xi \in [0, 1/2]$, by Lemma 3 we obtain that

$$\sum_j \alpha_j n_j^2 \le \sum_j \alpha_j \left(\frac{1+2\xi}{4} n_j^2 + (1-\xi)\, o_j^2\right)$$

which yields that the price of anarchy is

$$\frac{\sum_j \alpha_j n_j^2}{\sum_j \alpha_j o_j^2} \le \frac{4\,(1-\xi)}{3-2\xi}.$$

When $\xi \in [1/2, 1]$, by Lemma 4 we obtain that

$$\sum_j \alpha_j n_j^2 \le \sum_j \alpha_j \left(\xi n_j^2 + \frac{3-2\xi}{4} o_j^2\right)$$

which yields that the price of anarchy is

$$\frac{\sum_j \alpha_j n_j^2}{\sum_j \alpha_j o_j^2} \le \frac{3-2\xi}{4\,(1-\xi)}.$$

□

We note that when $\xi = 0$, i.e., for the case of totally selfish players this result implies the known 4/3 upper bound on the price of anarchy [11], while when $\xi = 1$, i.e., for completely altruistic players, the ratio is unbounded. Furthermore, as ξ increases from 0 to 1/2 the ratio improves from 4/3 to 1, and then deteriorates as ξ approaches 1; note that when $\xi = 0.7$ the ratio is again 4/3.

It is not hard to show that these bounds are tight. It suffices to consider a load balancing game with two machines with latency functions $f_1(x) = (2-\xi)\,x$ and $f_2(x) = (1-\xi)\,x$ and two players. Two assignments are equilibria in this setting: either assigning both players to the second machine (where the total latency is $4\,(1-\xi)$) or assigning one player at each machine (where the total latency is $3-2\xi$).

6 Extensions and Open Problems

In this paper, we have studied the impact of altruism on the system performance in atomic congestion games and have noticed that, surprisingly, altruism can be harmful in general. For the special case of symmetric load balancing games, we observe that altruism can be helpful in some cases; in particular, compared to selfishness, we have shown that altruism helps in decreasing the price of anarchy when $\xi \in [0, 0.7]$ but is harmful when $\xi \in (0.7, 1]$. We note that for $\xi = 1/2$, symmetric load balancing games with ξ-altruistic players admit only optimal solutions as equilibria.

Following [6], we have also briefly considered the case in which players are simultaneously selfish and spiteful (as opposed to altruistic). Similarly to the model in the current paper, we can define ξ-spiteful players for particular values of the parameter ξ. In this setting, player i aims to select a strategy $s \in S_i$ so that the quantity

$$\sum_{e \in s} \left(\alpha_e \left(n_e(A_{-i}, s) + \xi\right) + \beta_e \left(1 + \xi\right)\right)$$

is minimized given the strategies A_{-i} of the other players. This is equivalent to assuming that all players are selfish and each of them is forced to pay a tax equal to $\xi\alpha_e$ for each resource e she uses (this particular tax definition is called a universal tax function in [5]). Then, the cost of a player is the sum of her latency and the taxes she pays and the equilibria of the corresponding game are those assignments in which no player has an incentive to deviate in order to decrease her cost. Caragiannis et al. [5] have proved that the universal tax function with $\xi = \frac{3}{2}\sqrt{3} - 2 \approx 0.598$ yields the best possible price of anarchy which is equal to $1 + 2/\sqrt{3} \approx 2.155$. This result implies the rather surprising conclusion that ξ-spiteful behavior for the particular value of ξ leads to the best possible price of anarchy.

In our study herein, we have assumed that all players are unweighted, i.e., each controls a unit demand, and *homogeneous*, i.e., each player is ξ-altruistic (or ξ-spiteful) for the same value of ξ. It would be interesting to study the case of *heterogeneous* players with different behavior, i.e., each player i is ξ_i-altruistic (or ξ_i-spiteful). Furthermore, an interesting question from the system designer's point of view is whether the behavior of the players can be coordinated in order to always force them to reach efficient equilibria. Even in this case, one cannot hope to achieve a price of anarchy smaller than 2.012 in general. This value matches the tight bound on the price of anarchy of load balancing games with identical latency functions of the form $f(x) = x$ on all resources [4, 14]; in this case, any combination of selfish and altruistic or spiteful behavior of a player is actually equivalent to selfishness.

We plan to elaborate on the two claims above in the final version of the paper.

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