New Bounds on the Size of the Minim Feedback Vertex Set in Meshes and Butterflies*

IOANNIS CARAGIANNIS
Computer Technology Institute and University of Patras, Greece

CHRISTOS KAKLAMANIS
Computer Technology Institute and University of Patras, Greece

PANAGIOTIS KANELLOPOULOS
Computer Technology Institute and University of Patras, Greece

Abstract

Given a graph $G = (V, E)$, the minimum feedback vertex set S is a sub of vertices of minimum size, whose removal induces an acyclic subgra $G' = (V \backslash S, E')$. The problem of finding S is NP–complete in general grap although polynomial time solutions exist for particular classes of graphs. this paper we present upper and lower bounds on the size of the minimu feedback vertex set in meshes and butterflies improving results of Luc [10].

Keywords

Feedback Vertex Set, Mesh, Butterfly

1 Introduction

A *feedback vertex set* of a graph $G = (V, E)$ is a subset of vertices $S \subseteq$ removal from G induces an acyclic graph $G' = (V', E')$ with $V' = V \backslash S$ a $\{(u, v) \in E : u, v \in V'\}$. If the cardinality of S is the minimum possible, v a *minimum feedback vertex set* of G.

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The problem has been proved to be NP–complete [4], although pol time solutions have been found for particular classes of graphs, e.g., r graphs [12], cocomparability and convex bipartite graphs [8], cyclically r graphs [15], and interval graphs [9]. For general graphs, the best known mation algorithm has approximation ratio 2 and is due to Bafna *et al.* [1]

The problem has important applications to several different contexts. application was in combinatorial circuit design (see the discussion in [3 it also applies to operating system where resourse allocation mechanisn can prevent deadlocks are desirable. This can be carried out by finding a f vertex set in the dependence graph of the tasks that want to use a parti source and moving the corresponding tasks into a waiting queue so that dependence graph becomes acyclic (and deadlock–free) [15]. Clearly, in uation it is also important to move into the waiting queue the minimum number of tasks.

Other applications of the problem are in artificial intelligence (in the c satisfaction problem and Bayesian inference [2]), in the study of mono synchronous distributed systems [11], and, recently, in the problem of co placement in optical networks [5, 14].

1.1 Our results.

In this paper we give new bounds on the size of the minimum vertex set in and butterflies. We present simple alternative ways for computing small f vertex sets in 2–dimensional meshes. We also present a method which to meshes of higher dimensions. We prove that the bounds we obtain for are asymptotically optimal (provided that the size of the mesh is large) b almost matching lower bounds. Our bounds for meshes can also be ext tori.

For butterflies, we obtain our upper bound by improving the analysis gorithm presented in [10]. Using a simple argument, we obtain an almos ing lower bound, thus proving that our upper bound is asymptotically tigh dimension of the butterfly increases).

The best previously known bounds for the problem in the particular c graphs were due to Luccio [10]. In the following table, we summarize o results and relate them to previously known ones.

1.2 Outline of the paper.

The rest of the paper is structured as follows. In Section 2 we give a sir general lower bound argument which yields lower bounds in any graph. ternative upper bounds for the 2–dimensional mesh are presented in S The upper bound for meshes of higher dimensions is presented in Section the upper bound for the butterfly is given in Section 5.

	Previous results (Luccio [10])		Our results	
	Upper bound	Lower bound	Upper bound	Lower b
2-D mesh	$\frac{mn}{3}+\frac{m+n}{6}+o(m,n)$	$\frac{mn}{3}-\frac{n+m-2}{3}$	$\frac{mn}{3}-\frac{m+n-5}{6}$	
d-D mesh			$\frac{(d-1)n^d}{2d-1}$	$\frac{(d-1)n^d}{2d}$
Butterfly	$\frac{11d2^d}{32}+d$	$2^{d-1}\lfloor\frac{d+1}{2}\rfloor$	$\frac{(d+1/2)2^d}{3}$	$\frac{(d-1)2^d}{3}$

Table 1: Summary of our results.

2 Lower bounds

Luccio in [10] proved lower bounds for 2-dimensional meshes and tori, a bounds for butterfly networks. Her proofs are based on rather complex ar

In this section we present a simple argument useful to prove lowe in any network. Using this argument, we extend the known lower bo 2–dimensional meshes to meshes of higher dimensions and improve th lower bound for the size of the minimum feedback vertex set in the butte

Our proofs are based on the following observation.

Fact 1 *A set $S \subseteq V$ is a feedback vertex set of G if and only if the subgr induced by the vertices of $V \setminus S$ is a forest.*

Lemma 1 *Any feedback vertex set in a graph $G(V,E)$ with maximum has size at least*

$$\frac{|E|-|V|+1}{r-1}.$$

Proof. Let $S \subseteq V$ be a feedback vertex set of G. The subgraph $H(V',$ induced by the vertices in $V' = V \setminus S$ has $|E'| \geq |E| - r|S|$ edges. By Fact forest; thus, we obtain that

$$\begin{aligned} |V'| &\geq |E'|+1 \Rightarrow \\ |V|-|S| &\geq |E|-r|S|+1 \Rightarrow \\ |S| &\geq \frac{|E|-|V|+1}{r-1} \end{aligned}$$

We can immediately apply Lemma 1 to meshes, tori, butterflies, a networks. We only state the lower bounds for the d–dimensional mesh d–dimensional butterfly here.

Corollary 1 *The minimum feedback vertex set of the d–dimensional m $n \times n \times \ldots \times n = n^d$ nodes has size at least*

$$\frac{(d-1)n^d - dn^{d-1}+1}{2d-1}.$$

Corollary 2 *The minimum feedback vertex set of the d–dimensional butt size at least*

$$\frac{(d-1)2^d+1}{3}$$

Note that Corollary 2 improves the $2^{d-1}\lfloor\frac{d+1}{2}\rfloor$ lower bound presented in

A similar argument was recently used in [3] to prove a lower boun hypercube. We believe that using this method, we can achieve lower boun are close to the size of the minimum vertex set for graphs in which the degree is close to the maximum degree. Especially for meshes, we can u additional arguments to slightly improve the lower bound in Corollary $o(n^d)$ additive factor). Details are ommitted.

3 Upper bounds on the 2–dimensional mesh

For 2–dimensional meshes, Luccio in [10] shows a construction of a f vertex set on the $n \times m$ mesh which has size at most

$$\frac{mn}{3}+\frac{m+n}{6}+o(m,n).$$

The construction in [10] computes an optimal feedback vertex set in where $n = m = 2^r - 1$ for some integer $r > 0$.

Below, we show a way to improve the upper bound in [10]. We beg very simple construction.

Consider the partition of the nodes of a two dimensional mesh M in sets V_0, V_1, V_2 defined as

$$V_k = \{v_{ij} | i + j = k \bmod 3\}$$

for $k = 0, 1, 2$. We observe that any of the sets V_0, V_1, and V_2 is a feedbac set of $M = (V(M), E(M))$, since, for any $k = 0, 1, 2$, the subgraph of M by the vertices in $V(M)\backslash V_k$ consists of disjoint paths. Since the sets V_0, V disjoint, at least one of them has size at most $\lfloor |V(M)|/3 \rfloor$. We obtain the f result.

Theorem 1 *For any integers $n, m > 0$, the size of the minimum feedbac set of a $m \times n$ mesh is at most $\lfloor \frac{mn}{3} \rfloor$.*

An example of this construction is depicted in Figure 1. Now, we sho to improve Theorem 1.

Given a $n \times m$ mesh M, we apply the above idea to the lower right ($(m-2)$ submesh M'. Let $S(M')$ be the feedback vertex set of M', that co vertex-disjoint paths. By Theorem 1, we obtain

$$|S(M')| \leq \frac{(n-2)(m-2)}{3}.$$

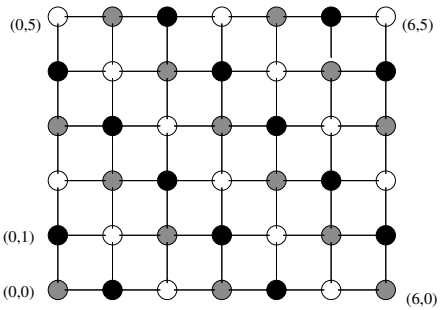

Figure 1: Partition of the nodes of the 6×7 mesh into three disjoint f
vertex sets.

Then, we consider the vertices in the boundary of M', i.e., the $n+m$
tices with coordinates

$$(1,0),(1,1),\ldots,(1,n-3),(1,n-2),(2,n-2),\ldots,(m-2,n-2),(m-1$$

Now we distinguish between three cases according to the feedback v
$S(M')$.

Case 1: $(2,n-3) \in S(M')$. In this case, we define the set of vertices S' to
the following vertices in the boundary:

$$(1,n-2),(3,n-2),\ldots,(n-1-(n \bmod 2),n-2)$$

and

$$(1,n-4),(1,n-6),\ldots,(1,n \bmod 2)$$

i.e., vertex $(1,n-2)$ and every second vertex on the same row and

Case 2: $(2,n-4) \in S(M')$. In this case, we define the set of vertices S' to
the following vertices in the boundary:

$$(1,n-2),(3,n-2),\ldots,(n-1-(n \bmod 2),n-2)$$

and

$$(1,n-3),(1,n-5),\ldots,(1,n+1 \bmod 2)$$

i.e., vertex $(1,n-2)$ and every second vertex on the same row, ar
$(1,n-3)$ and every second vertex downwards in the same column

Case 3: $(3,n-3) \in S(M')$. This case is symmetric to Case 2.

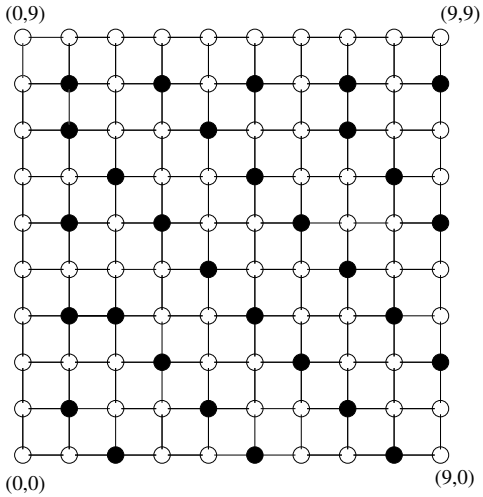

Figure 2: The feedback vertex set for the 10×10 mesh.

An example of this construction (for Case 2) is depicted in Figure 2.

It can be easily verified that

$$|S'| \leq \frac{n+m-1}{2}.$$

Now, S' is a feeedback vertex set for $M \backslash M'$, since there are no cycl
top 2 lines (respectively leftmost 2 columns).

The set $S(M') \cup S'$ is a feedback vertex set of M since its removal i
subgraph of M which has a path consisting of the vertices with coordinat

$$(0,0),(0,1),\ldots,(0,n-1),(1,n-1),(2,n-1),\ldots,(m-1,n-1)$$

connected to disjoint paths. Clearly, the induced subgraph is a tree.

Furthermore, we have that

$$\begin{aligned} |S(M') \cup S'| &\leq \frac{(n-2)(m-2)}{3} + \frac{n+m-1}{2} \\ &= \frac{nm}{3} - \frac{n+m-5}{6} \end{aligned}$$

Theorem 2 *For any integer $n, m > 0$, the size of the minimum feedback v
of a $m \times n$ mesh is at most*

$$\frac{nm}{3} - \frac{n+m-5}{6}.$$

4 Upper bound on meshes of higher dimensio

Our construction of feedback vertex sets of small size in d–dimensiona
($d \geq 2$) is based on the existence of distance–3 independent sets of large

Given a graph $G = (V,E)$, we call a set of nodes $T \subseteq V$ a *distance–3 dent set* if the distance between any two nodes in T is at least 3. An exa the 2–dimensional mesh is depicted in Figure 3.

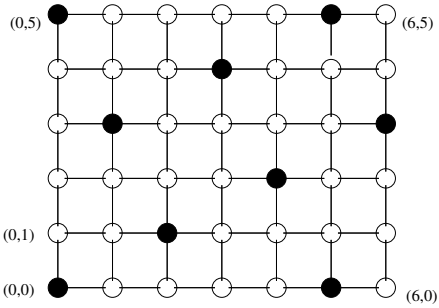

Figure 3: A distance–3 independent set on the 6×7 mesh.

Lemma 2 *In any d-dimensional mesh $n \times n \times \cdots \times n$, there exists a di independent set of size at least $\frac{n^d}{2d+1}$.*

Proof. Consider the partition of the n^d nodes of the mesh into $2d+1$ sets T_0, T_1, ..., T_{2d} such that set T_k contains the nodes with coordinates $(x_1, x$ such that

$$\sum_{i=1}^{d} i \cdot x_i = k \pmod{2d+1}.$$

Clearly, the sets $T_0, T_1, \ldots, T_{2d}$ are mutually disjoint. We will show t one of these sets is a distance–3 independent set of the mesh. In this way obtain the lemma.

Condider two nodes u and v in T_k (for some k such that $0 \leq k \leq$ coordinates $u = (x_1, x_2, \ldots, x_d)$ and $v = (y_1, y_2, \ldots, y_d)$, respectively. If u an in at least 3 coordinates, then the distance between u and v is at least 3.

Assume that u and v differ in two coordinates, i.e, $x_i \neq y_i$, $x_j \neq y_j$, and for $m \neq i, j$. Since u and v belong to T_k, it holds that

$$\sum_{i=1}^{d} i \cdot x_i = k \pmod{2d+1}$$

and

$$\sum_{i=1}^{d} i \cdot y_i = k \pmod{2d+1}.$$

We obtain that

$$i \cdot (x_i - y_i) + j \cdot (x_j - y_j) = 0 \pmod{2d+1}.$$

Now, if $|x_i - y_i| \geq 2$ or $|x_j - y_j| \geq 2$, the distance between u and v is a
Assume that $|x_i - y_i| = 1$ and $|x_j - y_j| = 1$ which yields $i \pm j = 0 \pmod$
a contradiction since $i \neq j$ and $1 \leq i, j \leq d$.

Assume that u and v differ in one coordinate, i.e., $x_i \neq y_i$, and x_m
$m \neq i$. Then,

$$i \cdot (x_i - y_i) = 0 \pmod{2d+1}.$$

Now, if $|x_i - y_i| = 1$ or $|x_i - y_i| = 2$, we obtain that $i > d$, a contradictio
the distance between u and v is at least 3. This completes the proof of the
□

Now, we will show how to construct a feedback vertex set in a d–dim
mesh using the construction of a distance–3 independent set presented in
2.

Theorem 3 *For any integers $n > 0, d \geq 2$, the minimum feedback vertex
d–dimensional mesh with $n \times n \times \ldots \times n = n^d$ nodes has size at most*

$$\frac{d-1}{2d-1} \cdot n^d.$$

Proof. Given a d–dimensional mesh G_d with $n \times n \times \ldots \times n = n^d$ no
construct a feedback vertex set as follows. We first apply Lemma 2 to
distance–3 independent set T in the $(d-1)$–dimensional mesh G_{d-1} of s

$$|T| \quad \geq \quad \frac{n^{d-1}}{2d-1}$$

Now, consider the set $V' \subseteq V$ of vertices which have the first $d-1$ coo
identical to the $d-1$ coordinates of some vertex in T. Clearly,

$$|V'| \quad = \quad n|T|$$

Consider the two (disjoint) subsets of $V \backslash V'$

$$S_1 = \left\{ v = (x_1, x_2, \ldots, x_d) \in V \backslash V' : \sum_{i=1}^{d} x_i = 0 \pmod 2 \right\}$$

and

$$S_2 = \left\{ v = (x_1, x_2, \ldots, x_d) \in V \backslash V' : \sum_{i=1}^{d} x_i = 1 \pmod 2 \right\}.$$

Since S_1 and S_2 are disjoint, it holds that

$$\min\{|S_1|,|S_2|\} \quad \leq \quad \frac{|V\backslash V'|}{2}$$

Using (1), (2), and (3), we obtain that

$$\min\{|S_1|,|S_2|\} \leq \frac{d-1}{2d-1}n^d.$$

It remains to prove that both S_1 and S_2 are feedback vertex sets in G. Consider the set S_1 (the proof for S_2 is similar). Removing the vertices in S_1 from G_d induces a graph H with a set of vertices $S_2 \cup V'$. Clearly, S_2 is an independent set of H, i.e., no two vertices of S_2 are adjacent. Furthermore, by construction, the vertices in V' consitute lines; a line for each vertex of G_{d-1} which belongs to V'. Thus, if there exists a cycle in H, then there must be at least a path in H between two vertices belonging to different lines. Note that, by our construction, the distance between two vertices v_1, v_2 belonging to different lines, is at least 3. Therefore, any path between v_1 and v_2 must contain vertices of both S_1 and S_2. This contradicts the assumption that H is the subgraph induced from G_d by removing the vertices in S_1. This completes the proof of the theorem.

The above statements can be slightly modified to show that the minimum feedback vertex set in a $n_1 \times n_2 \times \ldots \times n_d$ mesh has size at most

$$\frac{d-1}{2d-1}\prod_{i=1}^{d} n_i.$$

Furthermore, our techniques can also be applied to tori by increasing the size of the feedback vertex set by $O(n^{d-1})$. Details are ommitted.

5 Upper bound on the butterfly

In this section we present an asymptotically optimal upper bound on the size of the minimum feedback vertex set of a d–dimensional butterfly. Luccio and Pagli proved an upper bound of

$$(2^{d-2}+2^{d-4}+2^{d-5}+1)d = \frac{11d2^d}{32}+d.$$

This bound is $\Omega(d2^d)$ away from the lower bound we proved in Section 3 (Corollary 2). In what follows we prove that the algorithm presented in [ ] actually finds an asymptotically optimal feedback vertex set.

Recall that a d–dimensional butterfly is a graph $B_d = (V,E)$ comprising $(d+1)2^d$ vertices organized in $d+1$ levels of 2^d vertices each, where $v_{i,j}$

the j–th vertex at level i, with $0 \leq i \leq d$ and $0 \leq j \leq 2^d - 1$. For $i >$
connected with the two vertices $v_{i-1,j}$ and v_{i-1,j_i}, where j_i denotes th
whose binary representation differs from that of j in only the i–th bit pos

Algorithm BUTTERFLY-FVS

$S := \emptyset$
for $i := 1$ to d do
    for $j := 0$ to $2^d - 1$ do
        if $(v_{i-1,j} \notin S)$ and $(v_i, j \notin S)$ then $S := S \cup \{v_{i,j}\}$

Luccio in [10] proves that after the execution of algorithm BUTTERF
on a butterfly B_d, S is a feedback vertex set of B_d. The execution of the a
BUTTERFLY–FVS on a 4–dimensional butterfly is depicted in Figure 4.

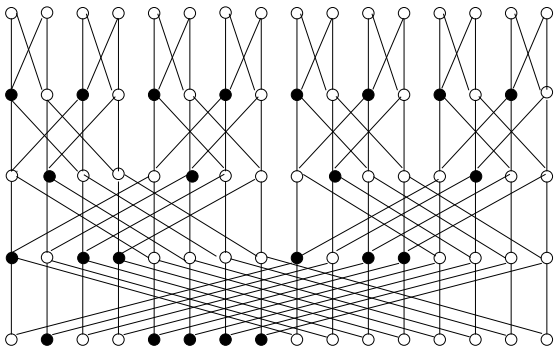

Figure 4: The execution of the algorithm on a 4–dimensional butterfly. B
tices are the vertices of the feedback vertex set S.

We denote by $b_i(d)$ the number of vertices of level i which belong in
F_d the size of S. Clearly,

$$b_0(d) = 0,$$

and since there are 2^d nodes at each level

$$b_i(d) = \frac{2^d - b_{i-1}(d)}{2}, \qquad 1 \leq i \leq d.$$

Using (4) and the fact that $b_0(d) = 0$ and $b_d(d) \leq 2^{d-1}$, we obtain th

$$\begin{aligned} F_d &= \sum_{i=0}^{d} b_i(d) \\ &= \sum_{i=1}^{d} \frac{2^d - b_{i-1}(d)}{2} \\ &= d2^{d-1} - \frac{\sum_{i=0}^{d-1} b_i(d)}{2} \\ &= d2^{d-1} - \frac{\sum_{i=0}^{d} b_i(d)}{2} + \frac{b_d(d)}{2} \\ &\leq d2^{d-1} - \frac{F_d}{2} + 2^{d-2} \Rightarrow \\ F_d &\leq \frac{(d+1/2)2^d}{3} \end{aligned}$$

Thus, we have proved the following

Theorem 4 *The minimum feedback vertex set of the d–dimensional butt size at most*

$$\frac{(d+1/2)2^d}{3}.$$

As mentioned in [10], algorithm BUTTERFLY–FVS can also be ap butterfly–like graphs (i.e., to toroidal butteflies, cube connected cycles). C ysis improves the known upper bounds on the size of the feedback vert these graphs as well.

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Ioannis Caragiannis is with Computer Technology Institute and the Computer ing and Informatics Department, University of Patras, 26500 Rio, Greece.
E-mail: caragian@cti.gr

Christos Kaklamanis is with Computer Technology Institute and the Compu neering and Informatics Department, University of Patras, 26500 Rio, Greece.
E-mail: kakl@cti.gr

Panagiotis Kanellopoulos is with Computer Technology Institute and the Com gineering and Informatics Department, University of Patras, 26500 Rio, Greece.
E-mail: kanellop@ceid.upatras.gr