A Logarithmic Approximation Algorithm for the Minimum Energy Consumption Broadcast Subgraph Problem ⋆

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Abstract

Motivated by the problem of supporting energy–efficient broadcasting in ad hoc wireless networks, we study the *Minimum Energy Consumption Broadcast Subgraph* (MECBS) problem. We present the first logarithmic approximation algorithm for the problem which uses an interesting reduction to Node–Weighted Connected Dominating Set.

Key words: graph algorithms, approximation algorithms, wireless networks

1 Introduction

Wireless networks have received significant attention during the recent years. Especially, *ad hoc wireless networks* emerged due to their potential applications in battlefield, emergency disaster relief, etc. [11,9]. Unlike traditional wired networks or cellular wireless networks, no wired backbone infrastructure is installed for ad hoc wireless networks.

A node (or station) in these networks is equipped with an omnidirectional antenna which is responsible for sending and receiving signals. Communication in these networks is established by assigning to each station a transmitting

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power. In the most common power attenuation model [9], the signal power falls as $1/r^\alpha$, where r is the distance from the transmitter and α is a constant which depends on the wireless environment (typical values of α are between 1 and 6). So, a transmitter can send a signal to a receiver if

$$\frac{P_s}{d(s,t)^\alpha} \geq \gamma$$

where P_s is the power of the transmitting signal, $d(s,t)$ is the euclidean distance between the transmitter and the receiver, and γ is the receiver's power threshold for signal detection which is usually normalized to 1.

So, communication from a node s to another node t may be established either directly if the two nodes are close enough and s uses adequate transmitting power, or by using intermediate nodes. Observe that due to the nonlinear power attenuation, relaying the signal between intermediate nodes may result in energy conservation.

A crucial issue in ad hoc wireless networks is to support communication patterns that are typical in traditional networks. These include broadcasting, multicasting, and gossiping (all–to–all communication). The important engineering question to be solved is to guarantee a desired communication pattern minimizing the total energy consumption. In this work, motivated by the problem of supporting energy–efficient broadcasting in ad hoc wireless networks, we study the *Minimum Energy Consumption Broadcast Subgraph* (MECBS) problem.

Consider a complete directed graph $G = (V, E)$ with a symmetric cost function $c : E \rightarrow R$ associated with its edges (i.e., $c(u,v) = c(v,u)$) and a special node $r \in V$. Given a weight assignment $w : V \rightarrow R$ to the nodes of G, the *transmission graph* G_w is the directed graph defined as follows. It has the same set of nodes with G and a (directed) edge (u,v) belongs to G_w if the weight assigned to node u is at least the cost of the edge (u,v), i.e., $w(u) \geq c(u,v)$. The Minimum Energy Consumption Broadcast Subgraph (MECBS) problem is to assign weights to the nodes of V so that for any node $u \in V - \{r\}$, the transmission graph G_w has a directed path from r to u. The objective is to minimize the sum of weights.

Note that by changing the connectivity requirements for the transmission graph, several new interesting combinatorial problems arise. Problems of this kind are examined in [1,4,8]; see also [3] for a recent survey. In these works, the objective is to establish a strongly connected transmission graph (with or without restrictions to its diameter). However, we may generalize the connectivity requirements for the transmission graph to introduce several new *wireless network design problems* in accordance to traditional network design problems [5] where we are given an edge–weighted graph and the main question is to establish a subgraph of minimum total edge–weight satisfying some connectivity requirements.

MECBS has been proved to be NP-hard and inapproximable within $(1-\epsilon)\ln n$ unless $\mathcal{NP} \subseteq DTIME\left(n^{O(\log\log n)}\right)$ (where n denotes the number of nodes in the graph) using approximation preserving reductions to Set Cover [2] and Connected Dominating Set [10]. However, to our knowledge, no logarithmic approximation algorithms for the problem were known prior to this work.

An important special case of MECBS (which reflects the properties of the engineering problem in practice) is when the nodes of the graph are points in the d–dimensional euclidean space and the cost function is defined as a α-th power of the distance function. Following the notation of [2], we denote this special case of the problem with MECBS[$\mathtt{N}_d^\alpha$]. Clementi et al. [2] have proved that MECBS[$\mathtt{N}_d^\alpha$] is $\mathcal{NP}$–hard for $d \geq 2$ and $\alpha > 1$. Wieselthier et al. [11] study the behaviour of several greedy algorithms for MECBS[$\mathtt{N}_2^2$]. They present experimental results for three simple algorithms: MST (Minimum Spanning Tree), BIP (Broadcast Incremental Power), and SPT (Shortest Path Tree). By exploring geometric properties of euclidean spanning trees, Clementi et al. [2] and Wan et al. [10] prove upper bounds on the approximation ratio of algorithm MST on instances of MECBS[$\mathtt{N}_2^\alpha$] for $\alpha \geq 2$. In [2], an upper bound of $\left(2\sqrt{5}\right)^\alpha$ is proved, while an improved bound of 12 is proved in [10]. Wan et al. [10] also prove lower bounds on the performance of algorithms MST, BIP, SPT, and BAIP (Broadcast Average Incremental Power), a greedy algorithm based on the well–known greedy algorithm for Set Cover. They show that even when instances of MECBS[$\mathtt{N}_2^2$] are considered, SPT and BAIP have approximation ratios $\Omega(n)$ and $\Omega(n/\log n)$, respectively. However, it is easy to verify that all the algorithms studied in [2,10,11] have approximation ratios $\Omega(n)$ when applied to general instances of MECBS.

In this work, we present the first logarithmic approximation algorithm for the problem which uses an interesting reduction to the *Node–Weighted Connected Dominating Set* (NWCDS) problem [6,7]. The Node–Weighted Connected Dominating Set problem is the following: Given a graph $G = (V, E)$ with weights on the nodes, find the smallest weighted subset S of nodes that induce a connected subgraph and each node in $V - S$ is adjacent in at least one node in S. For this problem, there exist $O(\log n)$–approximation algorithms, while sublogarithmic approximations are not feasible unless $P \subseteq DTIME[n^{O(\log\log n)}]$ [6,7].

The rest of the paper is structured as follows. In Section 2, we present the reduction of instances of MECBS to instances of NWCDS and show that using this reduction we can obtain logarithmic approximations for MECBS.

We conclude with open problems in Section 3.

2 A logarithmic approximation algorithm

In this section, we present a logarithmic approximation algorithm for MECBS. The algorithm uses a new reduction of instances of MECBS to instances of NWCDS.

First, we show how to transform any instance of MECBS to an instance of NWCDS. Let I_{MECBS} be an instance of MECBS which consists of a complete directed graph $G = (V, E)$ with $|V| = n$, a symmetric cost function $c : E \to R$, on the edges of G, and a special node $r \in V$. We will construct an instance I_{NWCDS} of the NWCDS which consists of a node–weighted undirected graph H.

We construct the graph H as follows. For each node v of G, the graph H has a set Z_v of n nodes $Z_{v,1}, Z_{v,2}, ..., Z_{v,n}$ which we call a *supernode*. The supernodes of H corresponding to different nodes of G are disjoint. Let $c_{v,i}$ be the i–th largest cost among the costs of the edges directed out of v in G. The weight associated with the node $Z_{v,i}$, $i = 1, ..., n-1$ of the supernode Z_v is set to $c_{v,i}$. The node $Z_{v,n}$ of the supernode Z_v has infinite weight.

The set of edges in H is defined as follows. Nodes $Z_{v,1}, ..., Z_{v,n-1}$ of the supernode Z_v form the complete graph K_{n-1} while node $Z_{v,n}$ is isolated from the other nodes of its supernode. Node $Z_{v,i}$, for $i = 1, ..., n-1$, is connected to all the nodes of those supernodes Z_u $(v \neq u)$ corresponding to nodes u of G which are connected to v by edges of cost no more than the weight of $Z_{v,i}$ (i.e., $c(v, u) \leq c_{v,i}$). The reduction is depicted in Figure 1.

Now, we will show that an approximate solution to NWCDS for instance I_{NWCDS} can be used to obtain an approximate solution to the MECBS for instance I_{MECBS} with similar approximation guarantee. We first show that the cost $OPT(I_{MECBS})$ of an optimal solution of I_{MECBS} is close to the cost $OPT(I_{NWCDS})$ of the optimal solution of I_{NWCDS}.

Lemma 1 $OPT(I_{NWCDS}) \leq 2 \cdot OPT(I_{MECBS})$.

PROOF. Consider a solution to MECBS for instance I_{MECBS} of minimum cost. Let w be the weight vector corresponding to this solution and G_w the transmission graph defined by w. Since the solution is optimal, the weight of each node v equals the cost of some edge directed out of v.

We construct a connected dominating set S for instance I_{NWCDS} as follows.

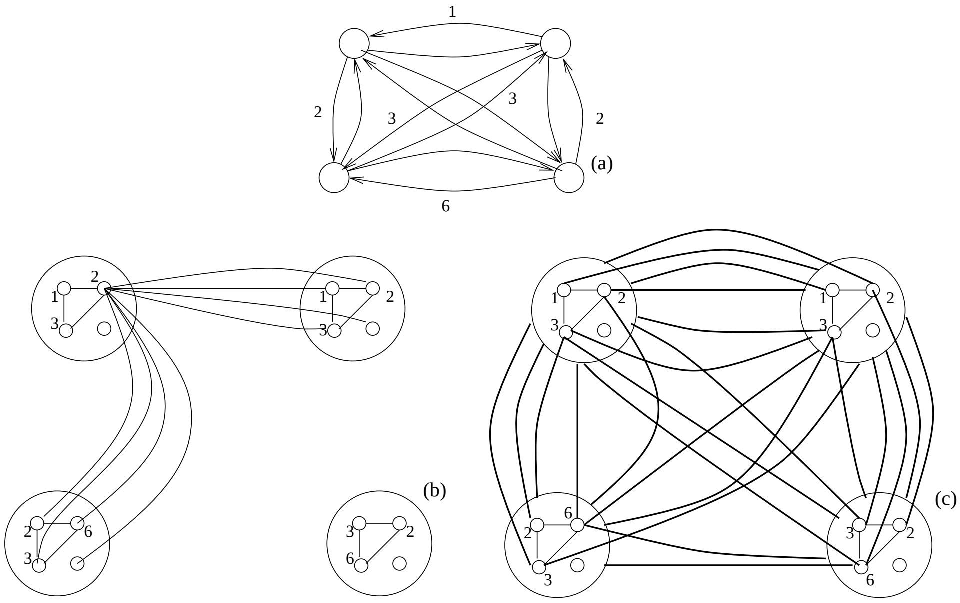

Fig. 1. The reduction to Node–Weighted Connected Dominating Set. (a) The graph G of an instance of MECBS. (b) The graph H of the corresponding instance of NWCDS. Each large cycle indicates a supernode. Only the edges incident to the node of weight 2 of the upper left supernode are shown. These edges are those which correspond to edges in (a) of cost at most 2, directed out of the left upper node. (c) The graph H of the corresponding instance of NWCDS. Thick edges connecting a node to a supernode represent edges connecting a node to all the nodes of a supernode.

For each node v of G_w which has at least one outgoing edge, we add to S the node $Z_{v,i(v)}$ of the supernode Z_v which has weight $w(Z_{v,i(v)}) = w(v) = \max_{(v,u)\in G_w}\{c(v,u)\}$.

Consider the induced subgraph G'_w of G_w which contains the nodes of G_w having at least one outgoing edge in G_w. By construction, each one of the nodes in S belongs to a different supernode of H mapping to a node in G'_w. Also, a directed edge between two nodes u and v in G'_w exists if and only if $w(u) \geq c(u,v)$. This means that the weight of the node $Z_{u,i(u)}$ is at least $w(u)$, and, thus, it is connected with node $Z_{v,i(v)}$ in H. This holds for any directed edge of G'_w and implies that the subgraph of H induced by S is isomorphic to G'_w (if we consider directed edges as undirected ones), and, thus, it is connected.

Also, S dominates all nodes of H except (possibly) for $Z_{r,n}$. The existence of a directed edge (u,v) in G_w implies that the node $Z_{u,i(u)}$ dominates all nodes of Z_v in H. Furthermore, all nodes in Z_r but $Z_{r,n}$ are dominated by $Z_{r,i(r)}$.

If $Z_{r,n}$ is adjacent to a node in S, then S is clearly a connected dominating set in H. The cost of S is

$$\sum_{Z_{v,i(v)} \in S} w(Z_{v,i(v)}) = \sum_{v \in G_w} w(v) = OPT(I_{MECBS}).$$

Now, if $Z_{r,n}$ is not adjacent to any node of S, we consider a node u such that the directed edge (r,u) exists in G_w. Let $Z_{u,j}$ be the node of Z_u with weight $c(u,r)$. Note that $Z_{u,j}$ is adjacent to both $Z_{u,i(u)}$ and $Z_{r,n}$. Thus, $S \cup \{Z_{u,j}\}$ is a connected dominating set in H. The cost of an optimal solution is at most the cost of $S \cup \{Z_{u,j}\}$ which is $OPT(I_{MECBS}) + w(Z_{u,j})$. Recall that $w(Z_{u,j}) = c(u,r) = c(r,u)$ and that $w(r) \geq c(r,u)$ since the edge (r,u) belongs to the transmission graph G_w. We obtain that

$$\begin{aligned} OPT(I_{NWCDS}) &\leq OPT(I_{MECBS}) + w(Z_{u,j}) \\ &= OPT(I_{MECBS}) + c(u,r) \\ &= OPT(I_{MECBS}) + c(r,u) \\ &\leq OPT(I_{MECBS}) + w(r) \\ &\leq 2 \cdot OPT(I_{MECBS}) \end{aligned}$$

The lemma follows. $\square$

We now show that a solution to NWCDS for instance I_{NWCDS} can give a solution to MECBS for instance I_{MECBS} of similar cost.

Lemma 2 *Given a solution to NWCDS for instance I_{NWCDS} of cost $COST(I_{NWCDS})$, we can construct in polynomial time a solution to MECBS for instance I_{MECBS} of cost $COST(I_{MECBS})$, such that*

$$COST(I_{MECBS}) \leq 2 \cdot COST(I_{NWCDS})$$

PROOF. Consider a connected dominating set S of H. We may assume that S contains at most one node from each supernode. If this is not the case and there are two nodes $Z_{u,i}$ and $Z_{u,j}$ $(i \neq j)$ in S with weights $w(Z_{u,i}) \geq w(Z_{u,j})$, we remove $Z_{u,j}$ from S. By repeatedly executing this procedure, we end up with a connected dominating set having at most one node per supernode. Furthermore, in any solution of finite cost, no node with infinite weight is contained in S. For every supernode Z_u which has a node $Z_{u,i(u)}$ contained in S, we associate a weight $w(Z_{u,i(u)})$ to node u of G. Nodes of G whose corresponding supernodes of H have no node contained in S are assigned zero weight.

Now, consider the transmission graph G_w defined by w. We will first show that the graph G_w is loosely connected, i.e., its undirected counterpart is connected.

Consider an edge $(Z_{u,i(u)}, Z_{v,i(v)})$ in H which connects two nodes of S. By the construction of graph H from G and the definition of the transmission graph G_w, we obtain that either (u,v) or (v,u) exists in G_w. This holds for any two nodes of S which are connected with an edge in H and, since the nodes of S are connected, this implies that the subgraph of G_w induced by the nodes corresponding to supernodes of H having a node in S is loosely connected. Also, observe that the nodes of any supernode Z_v containing no node in S are dominated by some node $Z_{u,i(u)}$ in S. By the construction of H from G and the definition of the transmission graph G_w, we obtain that for any node v of G_w with zero weight, there is a directed edge coming from a node with non–zero weight. We conclude that G_w is loosely connected.

We execute a spanning tree algorithm on G_w to compute a loosely connected spanning tree T_w of G_w. Next, we execute the following procedure that transforms T_w to an arborescence T'_w, i.e., a tree T'_w having a directed path from r to any other node. Starting from r, we compute a Breadth-First-Search (BFS) numbering of the nodes of T_w. We visit the nodes of T_w according to this BFS numbering and, at the step associated with a node u, we transform ingoing edges coming to u from nodes with numbers greater than u's to outgoing ones.

Now, by assigning weights $w'(u) = \max_{(u,v)\in T'_w} c(u,v)$ to the nodes of G, we obtain a transmission graph which contains T'_w as a subgraph and, thus, is a feasible solution to MECBS. An example of the construction of a transmission graph from a connected dominating set is depicted in Figure 2.

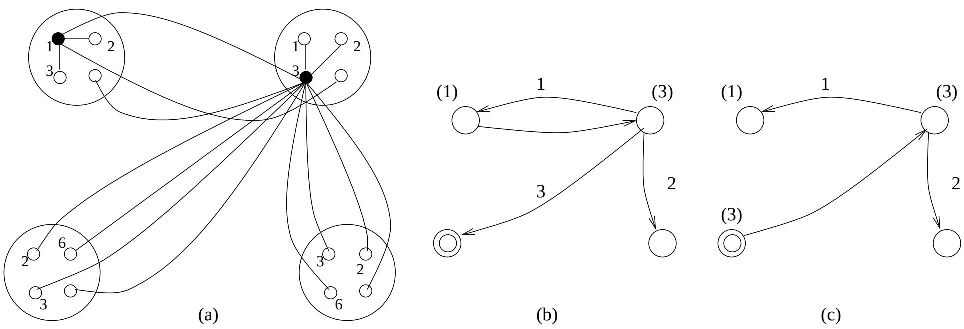

Fig. 2. An example of the reduction in proof of Lemma 2. Black nodes in (a) denote a connected dominating set. (b) The loosely connected transmission graph. (c) The final transmission graph. The lower left node in (b) and (c) is the special node r.

Let l_u be the node such that the edge (u, l_u) belongs to T'_w and has the maximum cost among all edges (u,v) directed out of u in T'_w. We call this node the *heavy child* of u. The transmission graph G_w contains either (u, l_u) or (l_u, u). If it contains (u, l_u), then $w(u) \geq c(u, l_u)$, otherwise $w(l_u) \geq c(l_u, u) = c(u, l_u)$. In any case,

$$w'(u) = \max_{(u,v)\in T'_w} c(u, l_u) \leq \max\{w(u), w(l_u)\} \leq w(u) + w(l_u).$$

The cost of the solution w' to MECBS is

$$\begin{aligned} COST(I_{MECBS}) &= \sum_{u \in V} w'(u) \\ &\leq \sum_{u \in V} (w(u) + w(l_u)) \\ &= \sum_{u \in V} w(u) + \sum_{u \in V} w(l_u) \\ &\leq 2 \cdot \sum_{u \in V} w(u) \\ &= 2 \cdot COST(I_{NWCDS}) \end{aligned}$$

Note that a node may be the heavy child of at most one node. Thus, each term of $\sum_{u \in V} w(l_u)$ appears only once, so $\sum_{u \in V} w(l_u) \leq \sum_{u \in V} w(u)$. This completes the proof of the lemma. □

Given an instance I_{MECBS} of MECBS, we first transform it to the corresponding instance I_{NWCDS} as described above. Then, we run an algorithm for the Node–Weighted Connected Dominating Set problem and use the technique described in the proof of Lemma 2 to construct a solution to the original problem. Using Lemmas 1 and 2, we can show that, given a ρ–approximate solution to NWCDS for instance I_{NWCDS} of cost $COST(I_{NWCDS})$, we can obtain a solution to MECBS for instance I_{MECBS} with cost $COST(I_{MECBS})$ which is within 4ρ of optimal. Indeed, by Lemma 1, we have

$$OPT(I_{MECBS}) \geq \frac{OPT(I_{NWCDS})}{2},$$

while Lemma 2 yields

$$\begin{aligned} COST(I_{MECBS}) &\leq 2 \cdot COST(I_{NWCDS}) \\ &\leq 2\rho \cdot OPT(I_{NWCDS}). \end{aligned}$$

Thus, the approximation ratio we obtain is

$$\frac{COST(I_{MECBS})}{OPT(I_{MECBS})} \leq 4\rho.$$

In [7], Guha and Khuller present a $1.35 \ln n$–approximation algorithm for the Node–Weighted Connected Dominating Set problem, where n is the number of nodes in the graph. Given an instance I_{MECBS} of MECBS with n nodes, the corresponding instance I_{NWCDS} has n^2 nodes. Thus, the cost of the solution of I_{NWCDS} is within $4 \cdot 1.35 \ln(n^2) = 10.8 \ln n$ of the optimal solution. The next theorem summarizes the discussion of this section.

Theorem 3 *There exists a* $10.8 \ln n$*–approximation algorithm for MECBS.*

Note that we have made no particular attempt to optimize the constants in the statement above. Possibly, slight modifications to our reduction may improve these constants.

3 Extensions and open problems

In this paper we consider symmetric instances of MECBS, i.e., instances in which any two opposite directed edges have the same cost. It is interesting to design a logarithmic approximation algorithm for the asymmetric problem. We may also define the "metric" α–MECBS problem, where distances satisfy the triangle inequality and the cost function is defined as the α–th power of the distance. Whether there are constant approximations (for constant α) in this case, is an interesting open problem. Furthermore, many interesting wireless network design problems arise by considering different connectivity requirements for the transmission graph. Problems of this kind reserve some attention.

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